New integrable PDEs of boomeronic type

2006 J. Phys. A: Math. Gen. 39 8349

(http://iopscience.iop.org/0305-4470/39/26/006)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 8349–8376 doi:10.1088/0305-4470/39/26/006

New integrable PDEs of boomeronic type

F Calogero[1] **and A Degasperis**[2]

[1] Dipartimento di Fisica, Università di Roma 'La Sapienza', 00185 Roma, Italy
[2] Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Roma, Italy

E-mail: francesco.calogero@roma1.infn.it and antonio.degasperis@roma1.infn.it

Received 7 March 2006, in final form 8 May 2006
Published 14 June 2006
Online at stacks.iop.org/JPhysA/39/8349

Abstract

Novel *integrable* systems of coupled *first-order* autonomous PDEs in 1 + 1 dimensions (space x and time t) are presented. Integrable covariant 2-vector and 3-vector PDEs, as well as higher-order integrable PDEs for a single, or a couple, of scalar-dependent variables (including an extension of the *sine-Gordon* equation and a remarkably neat, highly nonlinear *third-order* PDE), are also obtained by appropriate reductions of the basic matrix equations. The Lax pairs that characterize the integrable character of the basic matrix PDEs are also exhibited, as well as their single-soliton solutions. These solitons generally exhibit the *boomeronic* (and, less generically, the *trapponic*) phenomenology, namely they do not move uniformly, but rather (in an appropriate reference system) come in from one end in the remote past and *boomerang* back to that same end in the remote future (*boomerons*), or are *trapped* to oscillate around a value fixed by the initial data (*trappons*).

PACS numbers: 02.30.Ik, 02.30.Jr

1. Introduction

In a previous paper [5] we reported several novel integrable systems of coupled evolution PDEs of *nonlinear Schrödinger* type (in 1 + 1 dimensions, space x and time t) featuring *solitons* moving with time-dependent velocity: displaying typically, in an appropriate reference frame, the *boomeronic* phenomenology [2, 3], namely, even when single, coming in the remote past from one side and boomeranging back to that side with the same speed in the remote future; and occasionally displaying instead the *trapponic* phenomenology, namely, even when single, being trapped to oscillate around a fixed point (determined by the initial data). In the present paper we report analogous (to the best of our knowledge, also novel) results for a class of integrable autonomous nonlinear coupled evolution PDEs in (1 + 1) dimensions that are obtained in the standard way from a ('Lax') pair of linear matrix ODEs the coefficients of which are *first-degree* polynomials in the spectral parameter. This entails that the system of integrable PDEs discussed herein features only *first* derivatives both with respect to the time variable t (as the PDEs of *nonlinear Schrödinger* type) and as well with respect to the space variable x (in contrast to the PDEs of *nonlinear Schrödinger* type, that are of *second* order with respect to the space variable x—of course this characterization refers to the main dependent variables and to the basic structure of the coupled evolution PDEs under consideration, which are of course susceptible of being rewritten as higher-order equations via the elimination of some of the dependent variables). Some of the (systems of) PDEs we consider below are obtainable by merely eliminating certain terms in the (systems of) PDEs already considered in [5]; but there also are *additional* integrable (systems of) PDEs not obtainable in this manner, because some *reductions* which are permissible in the present context are not permissible in the context of the (systems of) PDEs of nonlinear Schrödinger type treated in [5]. Our main purpose in this paper—as indeed in [5]—is to display several integrable nonlinear (systems of) evolution PDEs in a manner as user-friendly as possible, having in mind the possibility that these findings be eventually utilized by researchers mainly interested, and knowledgeable, on using them in *applicative* contexts. We also display the single-soliton solutions of the basic matrix equations, as well as the Lax pair that provides the basic structure subtending the integrability of these matrix PDEs. Of course, as in the case of the results of [5], the availability of the integrability machinery for the evolution equations treated herein entails the possibility of obtaining via standard techniques many other relevant results, such as conservation laws, Hamiltonian structures, Bäcklund transformations, and so on; and moreover to exhibit the single-soliton and multi-soliton solutions for reduced evolution equations and to perform detailed analyses of their shapes and behaviours. Such results shall perhaps be provided in subsequent papers, especially for equations that turn out to be of theoretical and/or applicative interest.

In the following section 2 we report the evolution equations of the class indicated above; in the following section 3 we report, and tersely discuss, their single-soliton solutions; in section 4 we tersely report the Lax pair that underlies the integrable character of the systems of PDEs considered in this paper; and finally in section 5 we offer some final remarks. To avoid excessive repetitions, we generally assume the diligent reader to be familiar with the introductory remarks, the basic results and the notation of [5], referring hereafter to the formula (x) of that paper as (I.x). But we also cater to the casual reader who is only interested in browsing through the evolution equations covered by our treatment, so we display below some representative examples of them even when we might just mention how they can be obtained from the analogous (more general) equations reported in [5]; indeed we make an effort, by providing below a definition of all the notation we employ (albeit with some quite rare exceptions, when we refer the reader back to [5]), to make the present paper as completely self-contained as possible, while also respecting the imperative to avoid unnecessary repetitions.

Let us finally mention that a special case of the integrable system of coupled *autonomous* first-order PDEs treated in this paper is the well-known one [10, 11] describing the nonlinear interaction of *three* resonant waves. In that case the approach described in [5] and herein yields a novel class of solitonic solutions of that system. These results were considered sufficiently remarkable to be singled out and reported separately [6].

We end this introductory section by listing below the number of some of the equations that might be immediately looked at by the hasty browser who wishes to get an idea of the integrable coupled PDEs identified in this paper (note that we strived, perhaps at the cost of some repetitions, to present these results so as to allow the alert reader to jump immediately to these equations and understand their significance, including the comments that follow them—for instance, concerning their *dispersive*, or *nondispersive*, character, whenever we do include such a discussion): (40), (41), (53), (54), (61), (75), (78), (83), (86), (89), (91), (92), (100), (102), (106), (109), (114), (118), (126). And we exhibit here a few integrable PDEs, among those obtained below by additional reductions, that we deem particularly interesting (hereafter superimposed arrows denote 3-vectors, underlined symbols denote 2-vectors, and subscripted independent variables denote partial differentiations):

- *the boomeron equation* (see (35): here $\varphi(x,t) \equiv \rho_x(x,t)$ and $\vec{u}(x,t) \equiv \vec{v}_x(x,t)$ are the dependent variables and the two 3-vectors $\vec{a}$ and $\vec{b}$ are constant)

$$\varphi_t = \vec{b}\cdot\vec{u}_x, \qquad \vec{u}_t = \vec{a}\wedge\vec{u} + \varphi_x\vec{b} + \vec{z}\wedge\vec{u}, \qquad \vec{z}_x = \vec{b}\wedge\vec{u}, \tag{1a}$$

$$\rho_t = \vec{b}\cdot\vec{v}_x, \qquad \vec{v}_{xt} = \vec{a}\wedge\vec{v}_x + \rho_{xx}\vec{b} + (\vec{b}\wedge\vec{v})\wedge\vec{v}_x, \tag{1b}$$

- *the modified boomeron equation* (see (81): here $\varphi(x,t), \vec{u}(x,t)$ and $\vec{z}(x,t)$ are the dependent variables, the two 3-vectors $\vec{a}$ and $\vec{b}$ are constant and s is an arbitrary sign)

$$\varphi_t = \vec{b}\cdot\vec{u}_x, \qquad \vec{u}_t = \vec{a}\wedge\vec{u} + \varphi_x\vec{b} + \vec{z}\wedge\vec{u}, \qquad \vec{z}_x = s\varphi\vec{b}\wedge\vec{u}, \tag{2}$$

- *another covariant* 3-*vector PDE* (see (120): here $\vec{u}(x,t)$ and $\vec{z}(x,t)$ are the dependent variables, the two 3-vectors $\vec{a}$ and $\vec{b}$ are constant and s is an arbitrary sign)

$$\vec{u}_t = \vec{a}\wedge\vec{u} + (\vec{b}\cdot\vec{u}_x)\vec{b} - (\vec{b}\cdot\vec{b})\vec{u}_x + \vec{z}\wedge\vec{u}, \tag{3a}$$

$$\vec{z}_x = s(\vec{b}\cdot\vec{u})\vec{b}\wedge\vec{u}, \tag{3b}$$

- *the zoomeron* (or *Calapso*) *equation* (see (37): here $Z(x,t)$ is the dependent variable)

$$\left(\partial_t^2 - \partial_x^2\right)\left(\frac{Z_{xt}}{Z}\right) + (Z^2)_{xt} = 0, \tag{4}$$

- *the vector zoomeron* (or *vector Calapso*) *equation* (see (43): here the scalar $\varphi(x,t)$ and the 2-vector $\underline{u}(x,t)$ are the dependent variables)

$$\left(\partial_t^2 - \partial_x^2\right)\varphi = -\underline{u}\cdot\underline{u}, \qquad \underline{u}_{xt} = \underline{u}\varphi_{xt}, \tag{5}$$

- *another interesting integrable PDE* (see (86): here the *five* dependent variables are conveniently organized as a scalar, $\psi(x,t)$, and the *four* components of *two* 2-vectors, $\underline{v}(x,t)$ and $\underline{V}(x,t)$, and s is an arbitrary sign)

$$\left(\partial_t^2 - \partial_x^2\right)\psi = -\underline{v}\cdot\underline{V}, \qquad \underline{v}_t = \psi_t\underline{V}, \qquad \underline{V}_x = s\psi_x\underline{v}, \tag{6}$$

- *a highly nonlinear third-order PDE* (see (69): here $\Phi(x,t)$ is the dependent variable, and $D_n \equiv k_n\frac{\partial}{\partial t} + \omega_n\frac{\partial}{\partial x}$ are *three* linear differential operators, with k_n and ω_n being arbitrary constants)

$$(D_1D_2D_3\Phi)^2 = (D_1D_2\Phi)(D_2D_3\Phi)(D_3D_1\Phi), \tag{7}$$

  and *another avatar* (see (66)) of this PDE, which obtains via the position $\Psi(x,t) = D_n\Phi(x,t)$ and reads as follows (with n having a fixed value, $n = 1$ or $n = 2$ or $n = 3$ *mod* (3)):

$$D_n \log\left[\frac{(D_{n-1}D_{n+1}\Psi)^2}{(D_{n-1}\Psi)(D_{n+1}\Psi)}\right] = \frac{(D_{n-1}\Psi)(D_{n+1}\Psi)}{(D_{n-1}D_{n+1}\Psi)}, \tag{8}$$

- *a generalized sine-Gordon equation* (as a system of *two* coupled PDEs, see (96): here $\theta(x,t)$ and $\rho(x,t)$ are the *two* dependent variables, s is an arbitrary sign)

$$\theta_{tt} - \theta_{xx} + \tan(\theta)\rho_{tt} + \cot(\theta)\rho_{xx} = -\frac{1}{2}s\exp(2\rho)\sin(2\theta) + \frac{\theta_x\rho_x}{\sin^2(\theta)} - \frac{\theta_t\rho_t}{\cos^2(\theta)}, \tag{9a}$$

$$\rho_{xt} = -\theta_x\rho_t\tan(\theta) + \theta_t\rho_x\cot(\theta). \tag{9b}$$

We believe these integrable equations to be *new*, except for the *boomeron equation* (1) (see [2, 3]), the *zoomeron* (or *Calapso*) *equation* (4) (see [3]), the *vector zoomeron* (or *vector Calapso*) *equation* (5) (see [9]) and the *generalized sine-Gordon equation* (9) (which can be transformed into the classical isothermic system [9] via an appropriate transformation; we are grateful to W K Schief for pointing this out); as already mentioned, a more detailed investigation of some of them than that reported in this paper will be published separately if their theoretical and applicative interest will warrant it.

2. Integrable PDEs

The mother of all the evolution equations considered in this paper is the following integrable matrix PDE in 1 + 1 dimensions (see (I.4)):

$$Q_t - [C^{(0)}, Q] - \sigma[C^{(1)}, Q_x] = -\sigma\{W, Q\}, \tag{10a}$$

$$W_x = [C^{(1)}, Q^2]. \tag{10b}$$

Here and throughout we use the following standard notation for the commutator and the anticommutator:

$$[A, B] \equiv AB - BA, \qquad \{A, B\} \equiv AB + BA. \tag{11}$$

The main dependent variable $Q \equiv Q(x, t)$, the auxiliary dependent variable $W \equiv W(x, t)$ and the constant (space- and time-independent) quantities $C^{(0)}, C^{(1)}$ and σ are all $N \times N$ matrices (namely, square matrices with N lines and N columns), having the general block structure

$$\begin{pmatrix} N^{(+)} \times N^{(+)} & N^{(+)} \times N^{(-)} \\ N^{(-)} \times N^{(+)} & N^{(-)} \times N^{(-)} \end{pmatrix}, \tag{12}$$

where of course $N = N^{(+)} + N^{(-)}$ and $N^{(+)}, N^{(-)}$ are two, *a priori* arbitrary, positive integers. In particular $\sigma, C^{(0)}, C^{(1)}$ and $W(x, t)$ are block-diagonal, while $Q(x, t)$ is block-off-diagonal:

$$\sigma = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \tag{13}$$

$$C^{(j)} = \begin{pmatrix} C^{(j)(+)} & \mathbf{0} \\ \mathbf{0} & C^{(j)(-)} \end{pmatrix}, \qquad j = 1, 2, \tag{14}$$

$$Q = \begin{pmatrix} \mathbf{0} & Q^{(+)} \\ Q^{(-)} & \mathbf{0} \end{pmatrix}, \qquad W = \begin{pmatrix} W^{(+)} & \mathbf{0} \\ \mathbf{0} & W^{(-)} \end{pmatrix}. \tag{15}$$

Note that these definitions entail that the block-diagonal matrices $C^{(j)}$ and $W(x, t)$ *commute* with σ, while the block-off-diagonal matrix $Q(x, t)$ *anticommutes* with σ:

$$[C^{(j)}, \sigma] = [W, \sigma] = \{Q, \sigma\} = 0. \tag{16}$$

Hereafter, without significant loss of generality, we assume validity of the boundary condition

$$W(-\infty, t) = 0, \tag{17}$$

entailing (see (10))

$$\mathrm{trace}[W(x, t)] = 0. \tag{18}$$

This matrix evolution equation (10), being a *first-order* PDE, is the simplest one of the class of integrable matrix evolution equations introduced and discussed in [5]. As already mentioned above, we refer to this paper [5] for the basic analysis of the integrability of this class of nonlinear PDEs. Additional standard properties of this class of integrable PDEs, such as conservation laws, Hamiltonian structures, Bäcklund transformations, nonlinear superposition formulae, multisoliton solutions and so on, will perhaps be treated in subsequent papers. In the present paper we merely display the simpler integrable (systems of) PDEs belonging to the class under consideration here, see (10), as obtained by applying *four* standard *reductions* to this class of PDEs and by moreover restricting attention to *small* values of the integers $N^{(+)}$ and $N^{(-)}$ (and often also performing additional reductions, including some yielding *higher-order* PDEs); and for this class of matrix PDEs (without reductions) we exhibit and tersely discuss the corresponding single-soliton solution and also report the Lax pair that underlies its integrability. Our hope is that the following display of several, rather neat and presumably novel, examples of integrable systems of coupled PDEs constitute a useful service to the community of researchers interested in nonlinear phenomena, including in particular those focussed on applications.

Let us now report, for future reference, two general remarks.

Remark 2.1. Via the positions

$$\sigma C^{(1)} \equiv \tilde{C}, \qquad \sigma W(x,t) \equiv \tilde{W}(x,t), \tag{19}$$

the matrix PDE (10) can be rewritten in the following equivalent form:

$$Q_t + [Q, C^{(0)}] - \{Q_x, \tilde{C}\} = [Q, \tilde{W}], \tag{20a}$$

$$\tilde{W}_x = [\tilde{C}, Q^2]. \tag{20b}$$

Remark 2.2. Via the linear coordinate transformation

$$\tilde{x} = ax + bt, \qquad \tilde{t} = cx + dt \tag{21a}$$

with $ad - bc \neq 0$, entailing

$$\frac{\partial}{\partial t} = b\frac{\partial}{\partial \tilde{x}} + d\frac{\partial}{\partial \tilde{t}}, \qquad \frac{\partial}{\partial x} = a\frac{\partial}{\partial \tilde{x}} + c\frac{\partial}{\partial \tilde{t}}, \tag{21b}$$

the above PDEs, (20), become (here and in the following $\mathbf{1}$ denotes the unit matrix)

$$\left\{\frac{d}{2}\mathbf{1} - c\tilde{C}, Q_{\tilde{t}}\right\} + \left\{\frac{b}{2}\mathbf{1} - a\tilde{C}, Q_{\tilde{x}}\right\} = [Q, \tilde{W} - C^{(0)}], \tag{22a}$$

$$c\tilde{W}_{\tilde{t}} + a\tilde{W}_{\tilde{x}} = [\tilde{C}, Q^2]. \tag{22b}$$

Note that the matrix coefficient $C^{(0)}$ could be eliminated by replacing, in these two equations (22) $\tilde{W}$ with $\tilde{W} + C^{(0)}$. But we prefer to keep the constant matrix $C^{(0)}$ in (22), and throughout, and to impose instead the boundary condition

$$\tilde{W}(-\infty, t) = 0, \tag{23}$$

see (17) and (19).

Via (13), (14) and (15) the $N \times N$ matrix PDE (10) reads as follows:

$$Q_t^{(\pm)} - [C^{(0)(\pm)}Q^{(\pm)} - Q^{(\pm)}C^{(0)(\mp)}] \mp [C^{(1)(\pm)}Q_x^{(\pm)} - Q_x^{(\pm)}C^{(1)(\mp)}]$$
$$= \mp [W^{(\pm)}Q^{(\pm)} + Q^{(\pm)}W^{(\mp)}], \tag{24a}$$

$$W_x^{(\pm)} = [C^{(1)(\pm)}, Q^{(\pm)}Q^{(\mp)}]. \tag{24b}$$

Here of course $Q^{(-)} \equiv Q^{(-)}(x,t)$ is an $N^{(-)} \times N^{(+)}$ rectangular matrix, and likewise $Q^{(+)} \equiv Q^{(+)}(x,t)$ is an $N^{(+)} \times N^{(-)}$ rectangular matrix; $W^{(-)} \equiv W^{(-)}(x,t)$ is an $N^{(-)} \times N^{(-)}$

square matrix, and likewise $W^{(+)} \equiv W^{(+)}(x,t)$ is an $N^{(+)} \times N^{(+)}$ square matrix; $C^{(j)(\pm)}$ are, respectively, $N^{(+)} \times N^{(+)}$ and $N^{(-)} \times N^{(-)}$ *constant* square matrices. The (generally rectangular; but see below) matrices $Q^{(\pm)}(x,t)$ are the main dependent variables, and the reductions relate the matrix $Q^{(+)}(x,t)$ to the matrix $Q^{(-)}(x,t)$, which is hereafter denoted as follows:

$$Q^{(-)}(x,t) \equiv U(x,t). \tag{25}$$

The square matrices $W^{(\pm)}(x,t)$ are the auxiliary dependent variables. These matrices are defined by (24*b*), supplemented by the boundary condition (see (17))

$$W^{(\pm)}(-\infty,t) = 0, \tag{26a}$$

that, together with (24*b*), entails that these matrices are traceless (see (18)):

$$\mathrm{trace}[W^{(\pm)}(x,t)] = 0. \tag{26b}$$

To define the *reductions* it is convenient to introduce the following *diagonal* $N \times N$ matrix S, the block-diagonal structure of which reads

$$S = \begin{pmatrix} S^{(+)} & \mathbf{0} \\ \mathbf{0} & S^{(-)} \end{pmatrix}, \tag{27a}$$

with the two *diagonal* matrices $S^{(+)}$ and $S^{(-)}$ made up, without significant loss of generality, just of 'signs',

$$S^{(\pm)} = \mathrm{diag}\left(s_1^{(\pm)}, s_2^{(\pm)}, \ldots, s_{N^{(\pm)}}^{(\pm)}\right), \qquad \left[s_n^{(\pm)}\right]^2 = 1. \tag{27b}$$

Note that this definition of the *diagonal* $N \times N$ matrix S entails the condition

$$S^2 = \mathbf{1}. \tag{27c}$$

The *four* main types of reductions are characterized respectively by the *four* conditions (see (25))

$$Q^{(-)}(x,t) = U(x,t), \qquad Q^{(+)}(x,t) = \mathbf{1}, \tag{28a}$$

$$Q^{(-)}(x,t) = U(x,t), \qquad Q^{(+)}(x,t) = S^{(+)}U^{\dagger}(x,t)S^{(-)}, \tag{28b}$$

$$Q^{(-)}(x,t) = U(x,t), \qquad Q^{(+)}(x,t) = sU^{*}(x,t), \qquad s = \pm, \qquad s^2 = 1, \tag{28c}$$

$$Q^{(-)}(x,t) = U(x,t), \qquad Q^{(+)}(x,t) = S^{(+)}U^{T}(x,t)S^{(-)}. \tag{28d}$$

Here of course the $N^{(-)} \times N^{(+)}$ matrix $U(x,t)$ is the dependent variable of the equations we shall display (generally, for small values of $N^{(+)}$, componentwise), and the appended symbols $\dagger$, $*$ respectively T in the last three of these formulae denote Hermitian conjugation, complex conjugation respectively transposition. Note that the first and third conditions (28*a*) and (28*c*) are only applicable when the matrices $Q^{(\pm)}$, see (15), are *square* matrices, namely they require that $N^{(+)} = N^{(-)} \equiv M$. (Actually the first reduction is also applicable in the *rectangular* case $N^{(-)} \neq N^{(+)}$, but the corresponding results do not seem sufficiently interesting to warrant their display.)

In the following subsections we discuss these four main types of *reductions*, of course requiring that they be compatible with the evolution equation (10) (this shall entail appropriate restrictions on the constant $N \times N$ matrices $C^{(0)}$ and $C^{(1)}$). The first of these reductions (28*a*) transforms back the Zakharov–Shabat (matrix) spectral problem, see (144*a*) below, into the Schrödinger (matrix) spectral problem, hence the corresponding results were essentially covered in our original papers of almost three decades ago [2, 3]; therefore the treatment given below of this case is extremely terse, being included here mainly in order to allow some useful comparisons. The second and third of these *reductions*, (28*b*) and (28*c*), were already considered in the context of the PDEs of *nonlinear Schrödinger* type on which we focussed in [5]; hence their treatment can also be rather terse, see below, since the corresponding findings are essentially special cases of results already reported in [5], except for the possibility, of which we take advantage below, to introduce in the present context more additional reductions than were permissible in the case treated in [5]. The last *reduction* (28*d*) is instead new since it was not applicable to the evolution PDEs of nonlinear Schrödinger type treated in [5].

2.1. Reductions of first type

The first type of *reductions* (see (28*a*)) requires that $N^{(-)} = N^{(+)} = M$ and reads

$$Q^{(-)}(x,t) = U(x,t), \qquad Q^{(+)}(x,t) = \mathbf{1}, \qquad W^{(-)}(x,t) = -W^{(+)}(x,t) = W(x,t). \tag{29a}$$

It is applicable provided the constant $M \times M$ matrices $C^{(j)(\pm)}$ satisfy the restrictions

$$C^{(0)(-)} = C^{(0)(+)} = C^{(0)}, \qquad C^{(1)(-)} = -C^{(1)(+)} = C^{(1)}. \tag{29b}$$

The general integrable system of matrix PDEs yielded by this reduction reads

$$U_t - [C^{(0)}, U] + \{C^{(1)}, U_x\} = [W, U], \tag{30a}$$

$$W_x = [C^{(1)}, U]. \tag{30b}$$

To make contact with previous formulations [2, 3] we set

$$U(x,t) = V_x(x,t), \tag{31}$$

so that (30*a*) with (26*a*) entails

$$W = [C^{(1)}, V], \tag{32}$$

and one obtains thereby from (30*a*) a single PDE for the $M \times M$ matrix $V(x,t)$:

$$V_{xt} = [C^{(0)}, V_x] - \{C^{(1)}, V_{xx}\} + [[C^{(1)}, V], V_x]. \tag{33}$$

For $M = 2$ the standard position in terms of the three Pauli matrices σ_j,

$$V(x,t) = \frac{1}{2}[\rho(x,t) + \vec{v}(x,t) \cdot \vec{\sigma}], \qquad C^{(0)} = -\frac{i}{2}\vec{a} \cdot \vec{\sigma}, \qquad C^{(1)} = -\frac{1}{2}\vec{b} \cdot \vec{\sigma}, \tag{34}$$

whereby one introduces the two constant 3-vectors $\vec{a}$ and $\vec{b}$ and, as dependent variables, the scalar $\rho(x,t)$ and the 3-vector $\vec{v}(x,t)$, reformulates (33) as the following *boomeron equation* [2, 3]:

$$\rho_t = \vec{b} \cdot \vec{v}_x, \tag{35a}$$

$$\vec{v}_{xt} = \vec{a} \wedge \vec{v}_x + \rho_{xx}\vec{b} + (\vec{b} \wedge \vec{v}) \wedge \vec{v}_x. \tag{35b}$$

While we refer to the literature [2, 3] for further discussions of this *integrable* PDE, we outline below the derivation from this equation (35), in the case when the two constant 3-vectors $\vec{a}$ and $\vec{b}$ are orthogonal,

$$\vec{a} \cdot \vec{b} = 0, \tag{36}$$

of the so-called *zoomeron* (or *Calapso*) *equation*. Let us recall that this *integrable* PDE,

$$(\partial_t^2 - \partial_x^2)\left(\frac{Z_{xt}}{Z}\right) + (Z^2)_{xt} = 0, \tag{37}$$

which was called 'zoomeron' equation [3, 4] by linguistic analogy with the 'boomeron' equation because for the (scalar) dependent variable we used the notation $Z \equiv Z(x, t)$, was largely employed by us and by others to illustrate the peculiarities of its solitonic sector via a movie, produced in the early 1980s by Chris Eilbeck, that displayed some of its two-soliton solutions. It was subsequently noted that this PDE (37) had been introduced almost a century ago by Pasquale Calapso in the context of differential geometry (see [9], and the references quoted there).

To perform this derivation we set, consistently with (36),

$$\vec{a} = a\hat{a}, \qquad \vec{b} = b\hat{b}, \qquad \hat{c} = \hat{a} \wedge \hat{b}, \tag{38}$$

introducing thereby the three orthogonal unit 3-vectors $\hat{a}$, $\hat{b}$ and $\hat{c}$, and the two constant scalars a and b. We then set

$$\vec{v}(x, t) = \alpha(x, t)\hat{a} + \beta(x, t)\hat{b} + \gamma(x, t)\hat{c}, \tag{39}$$

and we thereby rewrite the boomeron equation (35) as the following system of *four* coupled first- and second-order PDEs:

$$\begin{gathered} \rho_t = b\beta_x, \qquad \alpha_{xt} = b\alpha\beta_x, \\ \beta_{xt} = -a\gamma_x + b\rho_{xx} - \tfrac{1}{2}b(\alpha^2 + \gamma^2)_x, \qquad \gamma_{xt} = a\beta_x + b\gamma\beta_x. \end{gathered} \tag{40}$$

We then set $a = 0$ (and also, for simplicity, $b = 1$), so that (40) becomes (after x-integration of the third of these PDEs)

$$\rho_t = \beta_x, \qquad \beta_t = \rho_x - \tfrac{1}{2}(\alpha^2 + \gamma^2), \qquad \alpha_{xt} = \alpha\beta_x, \qquad \gamma_{xt} = \gamma\beta_x. \tag{41}$$

We then set, consistently with the first of these four PDEs,

$$\rho(x, t) = \varphi_x(x, t), \qquad \beta(x, t) = \varphi_t(x, t), \tag{42}$$

and rewrite the other three PDEs (41) as follows:

$$\left(\partial_t^2 - \partial_x^2\right)\varphi = -\underline{u} \cdot \underline{u}, \qquad \underline{u}_{xt} = \underline{u}\varphi_{xt}, \tag{43}$$

where $\underline{u} = 2^{-1/2}(\alpha, \gamma)$ is a two-dimensional vector. This is the so-called *vector zoomeron* (or *vector Calapso*) *equation* [9]. If we moreover set (consistently with this system of PDEs, (43)) the dependent variable α to zero, $\alpha(x, t) = 0$, then this system of PDEs (43) takes the simpler form

$$\left(\partial_t^2 - \partial_x^2\right)\varphi = -\gamma^2, \qquad \gamma_{xt} = \gamma\varphi_{xt}, \tag{44}$$

from which we get the single fourth-order PDE

$$\left(\partial_t^2 - \partial_x^2\right)\left(\frac{\gamma_{xt}}{\gamma}\right) + (\gamma^2)_{xt} = 0, \tag{45}$$

which coincides with the *zoomeron* (or *Calapso*) *equation* (37) up to the notational identification $\gamma(x, t) \equiv Z(x, t)$.

2.2. Reductions of second type

The second type of *reductions* (see (28*b*)) reads (see (I.23) and (I.25))

$$Q^{(-)}(x, t) = U(x, t), \qquad Q^{(+)}(x, t) = S^{(+)}U^{\dagger}(x, t)S^{(-)}, \tag{46a}$$

$$W^{(\pm)}(x, t) = -S^{(\pm)}W^{(\pm)\dagger}(x, t)S^{(\pm)}. \tag{46b}$$

It is applicable provided the constant matrices $C^{(j)(\pm)}$ satisfy the restrictions (see (I.21))

$$C^{(j)(\pm)} = -(-)^j S^{(\pm)}C^{(j)(\pm)\dagger}S^{(\pm)}, \qquad j = 0, 1. \tag{46c}$$

The general system of matrix PDEs yielded by this reduction reads (see (I.24))

$$U_t - [C^{(0)(-)}U - UC^{(0)(+)}] + [C^{(1)(-)}U_x - U_x C^{(1)(+)}] = W^{(-)}U + UW^{(+)}, \tag{47a}$$

$$W_x^{(+)} = [C^{(1)(+)}, S^{(+)}U^\dagger S^{(-)}U], \tag{47b}$$

$$W_x^{(-)} = [C^{(1)(-)}, US^{(+)}U^\dagger S^{(-)}]. \tag{47c}$$

Let us now consider a few special cases, corresponding to specific choices of the two positive integers $N^{(+)}$ and $N^{(-)}$.

For $N^{(+)} = 1$ and $N^{(-)} = D$ the matrix evolution equations (47) can be formulated as follows:

$$\vec{u}_t - C^{(0)}\,\vec{u} + C^{(1)}\vec{u}_x = W\,\vec{u}, \tag{48a}$$

$$W_x = [C^{(1)},\ \vec{u}\,\vec{u}^\dagger S]. \tag{48b}$$

Here the (column) D-vector $\vec{u} \equiv \vec{u}(x,t)$, having the D components $u_1, \ldots, u_D$, is the main dependent variable, the Hermitian conjugate (row) D-vector $\vec{u}^\dagger \equiv \vec{u}^\dagger(x,t)$ has of course the D components $u_1^*, \ldots, u_D^*$, and the $D \times D$ matrix $W \equiv W(x,t)$ is the auxiliary variable, uniquely defined by the last equation (48*b*) together with the usual boundary condition, see (26). These evolution equations (48) obtain from those written above (47) via the assignments

$$U(x,t) = \vec{u}(x,t), \qquad W^{(-)}(x,t) = W(x,t), \qquad W^{(+)}(x,t) = 0, \tag{49a}$$

$$S^{(+)} = 1, S^{(-)} = S, \qquad C^{(j)(+)} = c^{(j)}, \qquad C^{(j)(-)} = C^{(j)} + c^{(j)} \tag{49b}$$

with $c^{(j)}$ being scalar constants, which imply that the two constant $D \times D$ matrices $C^{(j)}$ that appear in these evolution equations satisfy the 'Hermiticity conditions'

$$C^{(j)} = -(-)^j SC^{(j)\dagger}S, \qquad j = 0, 1, \tag{49c}$$

and the auxiliary dependent variable $W \equiv W(x,t)$ (which is also a $D \times D$ matrix) satisfy the analogous 'Hermiticity condition'

$$W = -SW^\dagger S. \tag{49d}$$

The constant $D \times D$ matrix S appearing in these evolution equations is diagonal and it is made up of (*a priori* arbitrary) signs:

$$S = \mathrm{diag}[s_1, \ldots, s_D], \qquad s_k^2 = 1, \qquad k = 1, \ldots, D, \tag{50}$$

and moreover, without significant loss of generality, we can assume the two constant $D \times D$ matrices $C^{(j)}$ to be traceless,

$$\mathrm{Trace}[C^{(j)}] = 0, \qquad j = 0, 1. \tag{51}$$

For $D = 2$, we make the assignments

$$\vec{u}(x,t) = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix}, \qquad W(x,t) = \begin{pmatrix} \mathrm{i}\,z(x,t) & -s_1 w(x,t) \\ s_2 w^*(x,t) & -\mathrm{i}\,z(x,t) \end{pmatrix}, \tag{52a}$$

$$S = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}, \qquad C^{(0)} = \begin{pmatrix} \mathrm{i}a & s_1\tilde{a} \\ -s_2\tilde{a}^* & -\mathrm{i}a \end{pmatrix}, \qquad C^{(1)} = \begin{pmatrix} -b & s_1\tilde{b} \\ s_2\tilde{b}^* & b \end{pmatrix}, \tag{52b}$$

which clearly satisfy the Hermiticity requirements (49), provided the two constants a and b, as well as the auxiliary dependent variable $z \equiv z(x,t)$, are *real*, $a = a^*, b = b^*$,

$z(x,t)=z^*(x,t)$ (as we hereafter assume), while the other two constants, $\tilde{a}$ and $\tilde{b}$, as well as the other auxiliary dependent variable $w\equiv w(x,t)$ need *not* be real. Thereby these evolution equations (48) take the following form:

$$u_{1,t}-\mathrm{i}\,au_1-s_1\tilde{a}u_2-bu_{1,x}+s_1\tilde{b}u_{2,x}=\mathrm{i}\,zu_1-s_1wu_2, \tag{53a}$$

$$u_{2,t}+\mathrm{i}\,au_2+s_2\tilde{a}^*u_1+bu_{2,x}+s_2\tilde{b}^*u_{1,x}=-\mathrm{i}\,zu_2+s_2w^*u_1, \tag{53b}$$

$$w_x=2bs_1s_2u_1u_2^*+\tilde{b}(s_1|u_1|^2-s_2|u_2|^2), \tag{53c}$$

$$z_x=2\,\mathrm{Im}\,[\tilde{b}u_1^*u_2], \tag{53d}$$

with the latter two equations supplemented by the boundary conditions

$$w(-\infty,t)=z(-\infty,t)=0. \tag{53e}$$

Note that, without significant loss of generality, one can set $a=0$ (since the corresponding terms in (53*a*) and (53*b*) can be eliminated via the transformation $u_j(x,t)\to u_j'(x,t)=u_j(x,t)\exp\left(\mathrm{i}\frac{a}{b}x\right)$). If one moreover sets $\tilde{b}=0$, which via (53*d*) and (53*e*) entails $z(x,t)=0$, this system of PDEs takes the simpler form

$$u_{1,t}-s_1\tilde{a}u_2-bu_{1,x}=-s_1wu_2, \tag{54a}$$

$$u_{2,t}+s_2\tilde{a}^*u_1+bu_{2,x}=s_2w^*u_1, \tag{54b}$$

$$w_x=2bs_1s_2u_1u_2^*. \tag{54c}$$

If one moreover sets $\tilde{a}=0$ (or eliminates this constant by introducing the dependent variable

$$\tilde{w}(x,t)\equiv w(x,t)-\tilde{a}, \tag{55}$$

renouncing thereby to the requirement that $\tilde{w}(x,t)$ vanish at $x=-\infty$), then this system of PDEs becomes essentially identical (via a transformation of type (21)) to the standard system of PDEs describing the resonant interaction of three waves, as discussed in [6] (and, in the *real* case, below).

For $\tilde{a}\neq 0$, this system is instead easily seen to be *dispersive*, namely to possess (in the linear limit obtained by setting $w(x,t)=0$) the solution

$$u_j(x,t)=A_j\exp\{\mathrm{i}[kx-\omega(k)t]\},\qquad j=1,2, \tag{56a}$$

with the 'relativistic' dispersion relation

$$\omega(k)=\pm\sqrt{b^2k^2+s_1s_2|\tilde{a}|^2}. \tag{56b}$$

Hence this system yields a dispersive deformation of the standard (nondispersive) system describing the resonant interaction of three waves [10, 11] (certain new solutions of which have been recently introduced in [6] and [8]). This deformation is mathematically rather trivial (since it corresponds just to a trivial change of dependent variable, see (55)), but it might present some phenomenological interest, and if we ascertain this to be the case we shall provide elsewhere a discussion of this system and its solutions.

Let us also note that, if $\tilde{a}$ is *real*, $\tilde{a}=\tilde{a}^*$, the system (54) also admits solutions in which the dependent variables $u_1(x,t)$, $u_2(x,t)$ and $w(x,t)$ are all three *real*.

The display of other special cases of the integrable system (53) is left as an easy exercise for the diligent reader, who might find some guidance by consulting the analogous treatment in [5]. We present however here, for the special case with $\tilde{a}=0$ and all the dependent variables *real* ($u_1^*=u_1$, $u_2^*=u_2$, $w^*=w$; and we also set $b=1$ for simplicity), the derivation of a *higher-order* equation for a single dependent variable. Actually we prefer to do this for the following *real integrable* system,

$$D_n u_n = g_n u_{n+1} u_{n+2}, \qquad n = 1, 2, 3 \quad \text{mod}\,(3). \tag{57a}$$

Here and throughout the *three* linear differential operators D_n are defined as follows:

$$D_n = k_n \frac{\partial}{\partial t} + \omega_n \frac{\partial}{\partial x}, \tag{57b}$$

with k_n and ω_n being arbitrary *real* constants and the three coupling constants g_n as well *real*. Note that this system appears more general than (54), but in fact can be reduced to it (with $\tilde{a} = 0$) via the linear coordinate transformation (21), possibly augmented by appropriate (constant) rescalings of the *three* dependent variables $u_n(x, t)$. Note that this system is just the *real* subcase of the *three-resonant-interacting-waves equation* treated in [6, 8].

Clearly this system of three coupled first-order PDEs (57) entails the equations

$$D_n u_n^2 = 2 g_n u, \qquad \text{with} \qquad u \equiv u_1 u_2 u_3, \tag{58}$$

and this implies the relations

$$g_n D_m u_m^2 = g_m D_n u_n^2. \tag{59}$$

This suggests introducing the three new dependent variables $q_n(x, t)$ by setting

$$[u_n(x, t)]^2 = g_n q_n(x, t), \tag{60}$$

so that these new dependent variables satisfy the PDEs

$$D_n q_n = 2g\sqrt{q_1 q_2 q_3}, \qquad n = 1, 2, 3, \tag{61a}$$

with

$$g = \sqrt{g_1 g_2 g_3}. \tag{61b}$$

The property $D_n q_n = D_m q_m$, which is implied by these PDEs (61*a*), suggests moreover to introduce the three new dependent variables $\psi_n(x, t)$ via the positions

$$q_n(x, t) = D_{n+1} \psi_{n+2}(x, t) = D_{n+2} \psi_{n+1}(x, t), \qquad n = 1, 2, 3 \quad \text{mod}\,(3), \tag{62}$$

and to obtain a PDE involving just one dependent variable.

There are two strategies to reach this goal. The first uses the two relations (implied by this formula, (62))

$$q_{m-1} = D_{m+1} \psi_m, \qquad q_{m+1} = D_{m-1} \psi_m \tag{63}$$

(where m is always defined mod (3) and it is fixed), as well as the formula

$$q_m = \frac{1}{4g^2} \frac{(D_{m-1} D_{m+1} \psi_m)^2}{(D_{m-1} \psi_m)(D_{m+1} \psi_m)} \tag{64}$$

implied by the two formulae (63) together with (61*a*) (used twice, once for $n = m - 1$ and once for $n = m + 1$). It is then a matter of trivial algebra, by inserting in (61*a*) these three expressions of the three quantities q_n in terms of ψ_m, to obtain a single third-order integrable PDE for the single quantity

$$\Psi = 4g^2 \psi_m, \tag{65}$$

namely

$$D_m \log\left[\frac{(D_{m-1} D_{m+1} \Psi)^2}{(D_{m-1} \Psi)(D_{m+1} \Psi)}\right] = \frac{4g^2 (D_{m-1} \Psi)(D_{m+1} \Psi)}{(D_{m-1} D_{m+1} \Psi)}. \tag{66}$$

Note that this way of reducing the original system (57) yields just this single scalar PDE, because the three different values the index m may take in (66) correspond merely to a shuffling of the arbitrary parameters k_n, ω_n and g_n (see (57b)).

The second strategy is to observe that the relations (62) entail the formulae

$$D_{n+1}\psi_{n-1} = D_{n-1}\psi_{n+1}, \tag{67a}$$

$$q_{n-1} = D_n\psi_{n+1}, \qquad q_{n+1} = D_n\psi_{n-1}, \tag{67b}$$

and that these formulae (where n is always defined mod (3)) indicate that it is consistent to introduce a new dependent variable $\varphi(x,t)$ such that

$$\psi_n(x,t) = D_n\varphi(x,t), \tag{68a}$$

$$q_n(x,t) = D_{n-1}D_{n+1}\varphi(x,t) \qquad n = 1, 2, 3 \quad \mathrm{mod}\,(3). \tag{68b}$$

And the insertion of these expressions of the three quantities $q_n(x,t)$ in (61a) yields the single integrable third-order PDE

$$(D_1D_2D_3\Phi)^2 = (D_1D_2\Phi)(D_2D_3\Phi)(D_3D_1\Phi), \tag{69}$$

for the dependent variable

$$\Phi(x,t) = 4g^2\varphi(x,t). \tag{70}$$

Note that this definition of the variable $\Phi(x,t)$ entails, via the relations (68), the following relations with the variable $\Psi(x,t)$, see (65):

$$\Psi(x,t) = D_m\Phi(x,t). \tag{71}$$

The diligent reader will verify the consistency of this relation with the PDEs, (66) respectively (69), satisfied by $\Psi(x,t)$ respectively by $\Phi(x,t)$.

Let us conclude this discussion by noting that these results hold as well if the definition (57b) of the three linear differential operators D_n is generalized to read

$$D_n = k_n\left(\frac{\partial}{\partial t} + \vec{v}_n\cdot\vec{\nabla}\right), \tag{72}$$

where the three constant 3-vectors $\vec{v}_n$ are arbitrarily assigned and $\vec{\nabla}$ is the standard three-dimensional gradient operator, $\vec{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. Moreover the *integrability* of the 'three-interacting-wave' (system of coupled) PDEs is as well known to hold even if the definition (57b) is replaced by the more general definition (72) [11].

Finally, let us return to the general system (47), to consider the special case with $N^{(+)} = N^{(-)} = 2$. It is then convenient to parameterize the various matrices via the three Pauli 2×2 matrices σ_j and the unit 2×2 matrix $\mathbf{1}$ (hereafter often omitted), by setting

$$S^{(+)} = \mathbf{1}, \qquad S^{(-)} = s\mathbf{1}, \tag{73a}$$

$$C^{(0)(\pm)} = \mathrm{i}\,\vec{c}^{\,(0)(\pm)}\cdot\vec{\sigma}, \qquad C^{(1)(\pm)} = \vec{c}^{\,(1)(\pm)}\cdot\vec{\sigma}, \tag{73b}$$

$$U = \tfrac{1}{2}(\varphi + \vec{u}\cdot\vec{\sigma}), \qquad W^{(\pm)} = \mathrm{i}\,\vec{w}^{\,(\pm)}\cdot\vec{\sigma}, \tag{73c}$$

where we took advantage of the possibility of restricting consideration to traceless matrices $C^{(j)(\pm)}$ and $W^{(\pm)}$. Here of course the superimposed arrows identify 3-vectors. Note that the Hermiticity properties of these matrices, see (49), imply that the *four* constant 3-vectors

$\vec{c}^{(j)(\pm)}$, as well as the *two* 'auxiliary dependent variable' 3-vectors $\vec{w}^{(\pm)} \equiv \vec{w}^{(\pm)}(x,t)$, are all real:

$$\vec{c}^{(j)(\pm)*} = \vec{c}^{(j)(\pm)}, \qquad \vec{w}^{(\pm)*}(x,t) = \vec{w}^{(\pm)}(x,t). \tag{74}$$

The evolution equations take then the following form of ten coupled first-order PDEs:

$$\varphi_t = -\mathrm{i}\,\vec{\gamma}^{(0)(-)} \cdot \vec{u} + \vec{\gamma}^{(1)(-)} \cdot \vec{u}_x + \mathrm{i}\,\vec{z}^{(+)} \cdot \vec{u}, \tag{75a}$$

$$\vec{u}_t + \mathrm{i}\varphi\,\vec{\gamma}^{(0)(-)} + \vec{\gamma}^{(0)(+)} \wedge \vec{u} - \varphi_x\,\vec{\gamma}^{(1)(-)} + \mathrm{i}\,\vec{\gamma}^{(1)(+)} \wedge \vec{u}_x = \mathrm{i}\varphi\,\vec{z}^{(+)} + \vec{z}^{(-)} \wedge \vec{u}, \tag{75b}$$

$$\vec{z}^{(\pm)}_x = s\{\vec{\gamma}^{(1)(\pm)} \wedge \mathrm{Re}(\varphi^*\,\vec{u}) + \mathrm{Im}[(\vec{\gamma}^{(1)(\mp)} \cdot \vec{u}^*)_{\to u}]\}. \tag{75c}$$

Here and throughout the symbol $\wedge$ denotes the standard *three-dimensional* vector product and the symbol $\cdot$ the standard *three-dimensional* scalar product. The last equation must of course be supplemented with the boundary conditions

$$\vec{z}^{(\pm)}(-\infty,t) = 0. \tag{75d}$$

Note that, in order to write these equations in a neater form, we introduced the following notational changes:

$$\vec{\gamma}^{(j)(\pm)} = \vec{c}^{(j)(+)} \pm \vec{c}^{(j)(-)}, \qquad j = 0, 1, \qquad \vec{z}^{(\pm)} = \vec{w}^{(+)} \pm \vec{w}^{(-)}. \tag{76}$$

Note that the reality properties (74) imply that the four constant 3-vectors $\vec{\gamma}^{(j)(\pm)}$, as well as the two 'auxiliary dependent variable' 3-vectors $\vec{z}^{(\pm)} \equiv \vec{z}^{(\pm)}(x,t)$, are all *real*:

$$\vec{\gamma}^{(j)(\pm)*} = \vec{\gamma}^{(j)(\pm)}, \qquad \vec{z}^{(\pm)*}(x,t) = \vec{z}^{(\pm)}(x,t). \tag{77}$$

These equations (75) are essentially special cases of previous results (see equations (34) of [5]; note that we changed the sign appearing in the right-hand side of (75*d*), which was misprinted in [5]).

An additional reduction of the evolution equations (75) is permissible if, say, the two 3-vectors $\vec{\gamma}^{(j)(-)}$ vanish, $\vec{\gamma}^{(0)(-)} = \vec{\gamma}^{(1)(-)} = 0$, since one can then set $\varphi = 0$, $\vec{z}^{(+)} = 0$, and, for notational convenience, $\vec{\gamma}^{(0)(+)} = \vec{\gamma}^{(0)} = \vec{\gamma}^{(0)*}$, $\vec{\gamma}^{(1)(+)} = \vec{\gamma}^{(1)} = \vec{\gamma}^{(1)*}$ as well as $\vec{z}^{(-)}(x,t) = \vec{z}(x,t) = [\vec{z}(x,t)]^*$, whereby the evolution equations (75) take the following simpler form:

$$\vec{u}_t + \vec{\gamma}^{(0)} \wedge \vec{u} + \mathrm{i}\,\vec{\gamma}^{(1)} \wedge \vec{u}_x = \vec{z} \wedge \vec{u}, \tag{78a}$$

$$\vec{z}_x = -s\,\mathrm{Im}[(\vec{\gamma}^{(1)} \cdot \vec{u}^*)\,\vec{u}]. \tag{78b}$$

The last equation must of course be supplemented with the boundary condition

$$\vec{z}(-\infty,t) = 0. \tag{78c}$$

Note that the last two of these equations clearly imply that the 3-vector $\vec{z} \equiv \vec{z}(x,t)$ is orthogonal to the constant 3-vector $\vec{\gamma}^{(1)}$:

$$\vec{\gamma}^{(1)} \cdot \vec{z}(x,t) = 0. \tag{78d}$$

This system of five coupled PDEs is easily seen to be *dispersive,* namely to admit (in the linear limit obtained by setting $z(x,t) = 0$) the solution

$$\vec{u}(x,t) = \vec{A}\exp\{\mathrm{i}[kx - \omega(k)t]\}, \tag{79a}$$

with the function $\omega(k)$ featuring three branches, a trivial one, $\omega(k) = 0$, and two generally dispersive ones,

$$\omega(k) = \pm|k\vec{\gamma}^{(1)} - \vec{\gamma}^{(0)}| = \pm\sqrt{(\vec{\gamma}^{(1)} \cdot \vec{\gamma}^{(1)})k^2 - 2k(\vec{\gamma}^{(1)} \cdot \vec{\gamma}^{(0)}) + (\vec{\gamma}^{(0)} \cdot \vec{\gamma}^{(0)})}. \tag{79b}$$

Note that if the two constant 3-vectors $\vec{\gamma}^{(0)}$ and $\vec{\gamma}^{(1)}$ are *orthogonal*, $\vec{\gamma}^{(0)} \cdot \vec{\gamma}^{(1)} = 0$, the dispersion relation (79*b*) has the standard 'relativistic' form, while if $\vec{\gamma}^{(0)}$ vanishes, $\vec{\gamma}^{(0)} = 0$, the relation (79*b*) ceases to be dispersive (while if $\vec{\gamma}^{(1)}$ vanishes, $\vec{\gamma}^{(1)} = 0$, the system (78) ceases to be nonlinear).

An alternative reduction of the evolution equations (75) is permissible if, say, the two 3-vectors $\vec{\gamma}^{(0)(-)}$ and $\vec{\gamma}^{(1)(+)}$ vanish, $\vec{\gamma}^{(0)(-)} = \vec{\gamma}^{(1)(+)} = 0$, since it is then consistent to assume that the 3-vector $\vec{z}^{(+)}(x,t)$ also vanishes, $\vec{z}^{(+)}(x,t) = 0$, and that the remaining dependent variables are all *real*:

$$[\varphi(x,t)]^* = \varphi(x,t), \qquad [\vec{u}(x,t)]^* = \vec{u}(x,t), \qquad [z^{(-)}(x,t)]^* = z^{(-)}(x,t) \equiv z(x,t). \tag{80}$$

Then the evolution equations (75) become the following (*real*) *modified boomeron equation*:

$$\varphi_t = \vec{b} \cdot \vec{u}_x, \tag{81a}$$

$$\vec{u}_t = \vec{a} \wedge \vec{u} + \varphi_x \vec{b} + \vec{z} \wedge \vec{u}, \tag{81b}$$

$$\vec{z}_x = s\varphi \vec{b} \wedge \vec{u}, \tag{81c}$$

where we set for notational simplicity $-\vec{\gamma}^{(0)(+)} = \vec{a} = \vec{a}^*, \vec{\gamma}^{(1)(-)} = \vec{b} = \vec{b}^*$. The last of these equations is always supplemented by the boundary condition (78*c*), which, together with (81*c*), entails that the 3-vector $\vec{z}(x,t)$ is orthogonal to the constant 3-vector $\vec{b}$,

$$\vec{b} \cdot \vec{z}(x,t) = 0. \tag{81d}$$

Additional reductions of this system of evolution equations (81) are possible (and interesting). They are obtained by assuming to begin with that the two 3-vectors $\vec{a}$ and $\vec{b}$ are orthogonal, $\vec{a} \cdot \vec{b} = 0$, and by then introducing the three orthogonal unit 3-vectors $\hat{a}$ (parallel to $\vec{a} \equiv a\hat{a}$), $\hat{b}$ (parallel to $\vec{b} \equiv b\hat{b}$) and $\hat{c} = \hat{a} \wedge \hat{b}$ (orthogonal to both $\vec{a}$ and $\vec{b}$), as well as the components of the *two* 3-vectors $\vec{u}(x,t)$ and $\vec{z}(x,t)$ along these three unit 3-vectors:

$$\vec{u}(x,t) = u_1(x,t)\hat{a} + u_2(x,t)\hat{b} + u_3(x,t)\hat{c}, \tag{82a}$$

$$\vec{z}(x,t) = z_1(x,t)\hat{a} + z_3(x,t)\hat{c}. \tag{82b}$$

Then the evolution equations (81) read as follows:

$$\varphi_t = bu_{2x}, \tag{83a}$$

$$u_{1t} = -u_2 z_3, \qquad u_{2t} = b\varphi_x - au_3 + u_1 z_3 - u_3 z_1, \qquad u_{3t} = au_2 + u_2 z_1, \tag{83b}$$

$$z_{1x} = sb\varphi u_3, \qquad z_{3x} = -sb\varphi u_1. \tag{83c}$$

A first reduction of this system is obtained by assuming that the constant a vanishes, $a = 0$, by setting (merely for notational simplicity) the constant b to unity, $b = 1$, by then introducing the new dependent variable $\psi(x,t)$ via the position (consistent with (83*a*))

$$\psi_x = \varphi, \qquad \psi_t = u_2, \tag{84}$$

and by introducing the 2-vectors $\underline{v}(x,t)$ and $\underline{V}(x,t)$ via the assignment

$$\underline{v} \equiv (-u_1, u_3), \qquad \underline{V} \equiv (z_3, z_1). \tag{85}$$

Thereby the evolution equations (83) take the following neat form:

$$\psi_{tt} - \psi_{xx} = -\underline{v} \cdot \underline{V}, \qquad \underline{v}_t = \psi_t \underline{V}, \qquad \underline{V}_x = s\psi_x \underline{v}. \tag{86}$$

The way this system is obtained from the modified boomeron equation (81) is analogous to that followed to reduce the boomeron equation (35) to the vector zoomeron (or vector Calapso) equation (43). This justifies calling it *modified vector zoomeron* (or *modified vector Calapso*) *equation*. This connection is moreover evidenced by displaying the *Miura-type transformation* relating these two PDEs, which reads

$$\underline{u} = \frac{1}{\sqrt{2}}(\epsilon \underline{v} + \underline{V}), \qquad \varphi_{xt} = \epsilon \psi_{xt} + s\psi_x \psi_t, \qquad \epsilon^2 = s. \tag{87}$$

Note that the assumption that the two 2-vectors $\underline{v}(x,t)$ and $\underline{V}(x,t)$ depend on the independent variables x and t only via the function $\psi(x,t)$, $\underline{v} = \underline{v}[\psi(x,t)]$, $\underline{V} = \underline{V}[\psi(x,t)]$ (which is consistent with these equations (86) but amounts of course to a reduction) is immediately seen to entail that the dependent variable $\psi(x,t)$ satisfies the *sine-Gordon* equation

$$\psi_{tt} - \psi_{xx} = K \sin(\eta\psi + \kappa), \tag{88}$$

where K and κ are two constants and $\eta^2 = -s$.

The above integrable evolution equation (86) is a *real* system of *five* scalar equations (actually, *one* PDE and *four* ODEs; and note that it does not feature any arbitrary constant; but of course some such constants may be reintroduced by appropriate rescalings and translations of the variables). It can be further reduced to a system of *three* scalar equations (*one* PDE and *two* ODEs) if one assumes that, say, the dependent variables $u_1(x,t)$ and $z_3(x,t)$ vanish, $u_1(x,t) = z_3(x,t) = 0$, an assignment which is clearly consistent with (86), see (85). The integrable evolution equation takes thereby the reduced form

$$\psi_{tt} - \psi_{xx} = -u_3 z_1, \qquad u_{3t} = \psi_t z_1, \qquad z_{1x} = s\psi_x u_3, \tag{89}$$

which is already known in the classical geometric context of the theory of isothermic surfaces [9].

Another avatar of this integrable system of three coupled equations is obtained by replacing the two dependent variables $u_3(x,t)$ and $z_1(x,t)$ with the two new dependent variables $\rho(x,t)$ and $\delta(x,t)$ via the assignment

$$u_3(x,t) = \exp[\rho(x,t)] \sin[\eta\psi(x,t) + \delta(x,t)], \tag{90a}$$

$$z_1(x,t) = \eta \exp[\rho(x,t)] \cos[\eta\psi(x,t) + \delta(x,t)], \tag{90b}$$

with $\eta^2 = -s$. There obtains the following equivalent version of the *integrable* system of three coupled equations (89):

$$\psi_{tt} - \psi_{xx} = -\tfrac{1}{2}\eta \exp(2\rho) \sin[2(\eta\psi + \delta)], \tag{91a}$$

$$\rho_t \sin(\eta\psi + \delta) + \delta_t \cos(\eta\psi + \delta) = 0, \tag{91b}$$

$$\rho_x \cos(\eta\psi + \delta) - \delta_x \sin(\eta\psi + \delta) = 0. \tag{91c}$$

This provides a generalization of the *sine-Gordon* equation, to which it clearly reduces if the two dependent functions $\rho(x,t)$ and $\delta(x,t)$ are assumed to be just constant, an assumption which is clearly consistent with (91a) and (91c) (indeed, it is *implied* by these two ODEs even if one assumes to begin with only that $\rho(x,t)$ and $\delta(x,t)$ are both functions of the same single variable, say $\rho(x,t) = \rho[y(x,t)]$ and $\delta(x,t) = \delta[y(x,t)]$).

The integrable system (89) of *three* coupled equations (one PDE and two ODEs) in three dependent variables can moreover be reduced to a system of *two* coupled PDEs in *two* dependent variables, in several ways. The most obvious ones are by solving either the first or the second equation for z_1 and inserting the resulting expression in the other two equations, or by solving either the first or the third equation for u_3 and inserting the resulting expression in the other two equations. One obtains thereby easily the following *four* avatars of this integrable evolution equation:

$$T(\Lambda_{tt} - \Lambda_{xx})_x - T_x(\Lambda_{tt} - \Lambda_{xx}) = T^3\Lambda_x, \qquad \Lambda_t(\Lambda_{tt} - \Lambda_{xx}) = -TT_t, \tag{92a}$$

$$T_t\Lambda_{xt} - \Lambda_t T_{xt} = T\Lambda_x\Lambda_t^2, \qquad \Lambda_t(\Lambda_{tt} - \Lambda_{xx}) = -TT_t, \tag{92b}$$

$$X(\Lambda_{tt} - \Lambda_{xx})_t - X_t(\Lambda_{tt} - \Lambda_{xx}) = -X^3\Lambda_t, \qquad \Lambda_x(\Lambda_{tt} - \Lambda_{xx}) = XX_x, \tag{92c}$$

$$X_x\Lambda_{xt} - \Lambda_x X_{xt} = X\Lambda_t\Lambda_x^2, \qquad \Lambda_x(\Lambda_{tt} - \Lambda_{xx}) = XX_x. \tag{92d}$$

To obtain these equations we used the following convenient redefinition of the dependent variables:

$$\Lambda(x,t) = \eta\psi(x,t), \qquad T(x,t) = \eta u_3(x,t), \qquad X(x,t) = z_1(x,t) \qquad \eta^2 = -s, \tag{93}$$

which has the merit to display that the last two versions (92*c*) and (92*d*) obtain from the first two (92*a*) and (92*b*) via the simultaneous exchange of independent and dependent variables $x \Leftrightarrow t$, $X \Leftrightarrow T$ (which reflects a corresponding symmetry property of the system (89)). Let us emphasize that these *four* systems of *two* coupled PDEs are *equivalent*, inasmuch as they are all obtained from the *same* system of three equations (one PDE and two ODEs) (89); note however that systems (92*a*) and (92*c*) include a *third-order* PDE and a *second-order* PDE, while systems (92*b*) and (92*d*) include two *second-order* PDEs.

Another, possibly more elegant, avatar involving again only *two* independent variables obtains from (91) via the assignment

$$\theta(x,t) = \eta\psi(x,t) + \delta(x,t), \tag{94}$$

and the observation that the two ODEs (91*b*) and (91*c*) then read

$$\delta_t = -\rho_t\tan(\theta), \qquad \delta_x = \rho_x\cot(\theta). \tag{95}$$

It is then easily seen that the system of three coupled equations (91) (actually, one PDE and two ODEs) can be reformulated as the following integrable system of *two* coupled PDEs:

$$\theta_{tt} - \theta_{xx} + \tan(\theta)\rho_{tt} + \cot(\theta)\rho_{xx} = -\frac{1}{2}s\exp(2\rho)\sin(2\theta) + \frac{\theta_x\rho_x}{\sin^2(\theta)} - \frac{\theta_t\rho_t}{\cos^2(\theta)}, \tag{96a}$$

$$\rho_{xt} = -\theta_x\rho_t\tan(\theta) + \theta_t\rho_x\cot(\theta). \tag{96b}$$

It is clear that this provides an integrable generalization of the sine-Gordon equation, to which it obviously reduces when the dependent variable $\rho(x,t)$ is replaced (clearly compatibly with this system) by just a constant.

2.3. Reductions of third type

The third type of reductions (see (28c)) is only applicable to square matrices ($N^{(+)} = N^{(-)} = M$) and it reads (see (I.38))

$$Q^{(-)}(x,t) = U(x,t), \qquad Q^{(+)}(x,t) = sU^*(x,t), \tag{97a}$$

$$W^{(-)}(x,t) = W(x,t), \qquad W^{(+)}(x,t) = -W^*(x,t), \tag{97b}$$

where again the scalar constant s can be, without loss of generality, just a sign, $s = \pm, s^2 = 1$. This reduction is applicable provided the four constant square $M \times M$ matrices $C^{(j)(\pm)}$ satisfy the conditions (see (I.39))

$$C^{(j)(-)} = C^{(j)}, \qquad C^{(j)(+)} = (-)^j C^{(j)*}, \qquad j = 0, 1. \tag{97c}$$

Note that these conditions imply no restriction on the two $M \times M$ matrices $C^{(j)}$.

The general integrable system of matrix PDEs yielded by this reduction reads (see (I.40))

$$U_t - [C^{(0)}U - UC^{(0)*}] + [C^{(1)}U_x + U_x C^{(1)*}] = [WU - UW^*], \tag{98a}$$

$$W_x = s[C^{(1)}, UU^*], \tag{98b}$$

of course always with (26).

In the spirit of identifying the simplest nontrivial equations contained in this class one can consider the special case of these evolution equations (98) with $M = 2$, by setting

$$U(x,t) = \begin{pmatrix} u_1(x,t) & v(x,t) \\ v(x,t) & u_2(x,t) \end{pmatrix}, \qquad W(x,t) = \begin{pmatrix} \mathrm{i}z(x,t) & w(x,t) \\ -w^*(x,t) & -\mathrm{i}z(x,t) \end{pmatrix}, \tag{99a}$$

$$C^{(0)} = \begin{pmatrix} \mathrm{i}a & \tilde{a} \\ -\tilde{a}^* & -\mathrm{i}a \end{pmatrix}, \qquad C^{(1)} = \begin{pmatrix} -b & \tilde{b} \\ \tilde{b}^* & b \end{pmatrix}, \tag{99b}$$

with the restriction that the two constants a and b be *real*, $a = a^*, b = b^*$, and also *real* be the auxiliary dependent variable $z(x,t)$, $z(x,t) = z^*(x,t)$. Then the evolution equations (98) take the following simple form:

$$u_{1,t} - 2\mathrm{i}\,au_1 - 2\tilde{a}v - 2bu_{1,x} + 2\tilde{b}v_x = 2\mathrm{i}\,zu_1 + 2wv, \tag{100a}$$

$$u_{2,t} + 2\mathrm{i}\,au_2 + 2\tilde{a}^*v + 2bu_{2,x} + 2\tilde{b}^*v_x = -2\mathrm{i}\,zu_2 - 2w^*v, \tag{100b}$$

$$v_t + \tilde{a}^*u_1 - \tilde{a}u_2 + \tilde{b}^*u_{1,x} + \tilde{b}u_{2,x} = -w^*u_1 + wu_2, \tag{100c}$$

$$w_x = -s[\tilde{b}(|u_1|^2 - |u_2|^2) + 2b(u_1v^* + u_2^*v)], \tag{100d}$$

$$z_x = 2s\,\mathrm{Im}\,[\tilde{b}(vu_1^* + v^*u_2)], \tag{100e}$$

of course with the last two equations supplemented by the boundary conditions (53*e*). These equations (100) are essentially special cases of previous results (see eqs. (42) of [5]; note that we changed the subscript in the second term in the left-hand side of (100*b*), which was misprinted in [5]).

Another possible reduction of the evolution equations (98) with $M = 2$ obtains by setting

$$U(x,t) = \begin{pmatrix} u(x,t) & v_1(x,t) \\ v_2(x,t) & u(x,t) \end{pmatrix}, \qquad W(x,t) = \begin{pmatrix} w(x,t) & \mathrm{i}\,z_1(x,t) \\ \mathrm{i}\,z_2(x,t) & -w(x,t) \end{pmatrix}, \tag{101a}$$

with the three (scalar) auxiliary functions $w(x,t)$, $z_1(x,t)$ and $z_2(x,t)$ all *real*, $w(x,t) = w^*(x,t)$, $z_1(x,t) = z_1^*(x,t)$, $z_2(x,t) = z_2^*(x,t)$ and

$$C^{(0)} = \begin{pmatrix} a & \mathrm{i}a_1 \\ \mathrm{i}a_2 & -a \end{pmatrix}, \qquad C^{(1)} = \begin{pmatrix} \mathrm{i}b & b_1 \\ b_2 & -\mathrm{i}b \end{pmatrix}, \tag{101b}$$

with the six constants a, a_1, a_2, b, b_1, b_2 as well all *real*, $a = a^*, a_1 = a_1^*, a_2 = a_2^*$, $b = b^*, b_1 = b_1^*, b_2 = b_2^*$. The corresponding evolution equations then read

$$u_t - \mathrm{i}\,a_2v_1 - \mathrm{i}\,a_1v_2 + b_2v_{1x} + b_1v_{2x} = \mathrm{i}(z_2v_1 + z_1v_2), \tag{102a}$$

$$v_{1t} - 2av_1 - 2\mathrm{i}a_1u + 2\mathrm{i}bv_{1x} + 2b_1u_x = 2(wv_1 + \mathrm{i}z_1u), \tag{102b}$$

$$v_{2t} + 2av_2 - 2\mathrm{i}a_2u - 2\mathrm{i}bv_{2x} + 2b_2u_x = 2(-wv_2 + \mathrm{i}z_2u), \tag{102c}$$

$$w_x = 2s\,\mathrm{Re}[(b_1v_2^* - b_2v_1^*)u], \tag{102d}$$

$$z_{1x} = 2s[2b\,\mathrm{Re}(uv_1^*) - b_1\,\mathrm{Im}(v_1v_2^*)], \tag{102e}$$

$$z_{2x} = -2s[2b\,\mathrm{Re}(uv_2^*) + b_2\,\mathrm{Im}(v_2v_1^*)]. \tag{102f}$$

The last three of these equations must of course be supplemented by the boundary conditions

$$w(-\infty, t) = z_1(-\infty, t) = z_2(-\infty, t) = 0. \tag{102g}$$

Let us mention that the dispersion relation associated with the linear part of this system of evolution PDEs (namely, the expression $\omega(k)$ such that each of the three dependent variables $u(x,t)$, $v_1(x,t)$ and $v_2(x,t)$ is proportional—with a t- and x-independent constant of proportionality—to $\exp\{\mathrm{i}[kx - \omega(k)t]\}$ in the 'plane-wave' solution of the *linear* part of this system, obtained by setting $w = z_1 = z_2 = 0$) has *three* branches, a trivial one, $\omega(k) = 0$, and two others,

$$\omega(k) = \pm 2\sqrt{(b_1b_2 - b^2)k^2 - (a_1b_2 + a_2b_1 - 2ab)k + a_1a_2 - a^2}, \tag{103a}$$

which are however *not* dispersive: indeed the requirement that this expression yields a *real* value of $\omega(k)$ for *all real* values of k imposes the three conditions

$$b_1b_2 > b^2, \qquad a_1a_2 > a^2, \qquad (b_1b_2 - b^2)(a_1a_2 - a^2) > \tfrac{1}{4}(a_1b_2 + a_2b_1 - 2ab)^2, \tag{103b}$$

and it can be shown that, if the first *two* of these *three* inequalities are satisfied, then the *third* one cannot hold.

2.4. Reductions of fourth type

The fourth type of reductions (see (28*d*)) reads

$$Q^{(-)}(x,t) = U(x,t), \qquad Q^{(+)}(x,t) = S^{(+)}U^T(x,t)S^{(-)}, \tag{104a}$$

$$W^{(\pm)}(x,t) = -S^{(\pm)}W^{(\pm)T}(x,t)S^{(\pm)}. \tag{104b}$$

It is applicable provided the constant matrices $C^{(j)(\pm)}$ satisfy the restrictions

$$C^{(j)(\pm)} = -(-)^j S^{(\pm)}C^{(j)(\pm)T}S^{(\pm)}, \qquad j = 0, 1. \tag{104c}$$

The general form of the corresponding integrable system of matrix PDEs reads (see (47))

$$U_t - [C^{(0)(-)}U - UC^{(0)(+)}] + [C^{(1)(-)}U_x - U_xC^{(1)(+)}] = W^{(-)}U + UW^{(+)}, \tag{105a}$$

$$W_x^{(+)} = [C^{(1)(+)}, S^{(+)}U^TS^{(-)}U], \tag{105b}$$

$$W_x^{(-)} = [C^{(1)(-)}, US^{(+)}U^TS^{(-)}]. \tag{105c}$$

Let us now consider a few special cases, corresponding to specific choices of the two positive integers $N^{(+)}$ and $N^{(-)}$.

For $N^{(+)} = 1$ and $N^{(-)} = D$ the matrix evolution equations (47) can be reformulated as follows (see (48)):

$$\vec{u}_t - C^{(0)}\,\vec{u} + C^{(1)}\vec{u}_x = W\,\vec{u}, \tag{106a}$$

$$W_x = [C^{(1)}, \vec{u}\,\vec{u}^T S]. \tag{106b}$$

Here the (column) D-vector $\vec{u} \equiv \vec{u}(x,t)$, having the D components $u_1, \ldots, u_D$, is the main dependent variable, the transposed (row) D-vector $\vec{u}^T \equiv \vec{u}^T(x,t)$ has of course the D components $u_1, \ldots, u_D$, and the $D \times D$ matrix $W \equiv W(x,t)$ is the auxiliary variable, uniquely defined by the last equation (106b) together with the usual boundary condition, see (26). These evolution equations (106) obtain from those written above (105) via the assignments

$$U = \vec{u}, \qquad W^{(-)} = W, \qquad W^{(+)} = 0, \tag{107a}$$

$$S^{(+)} = 1, \qquad S^{(-)} = S, \qquad C^{(j)(+)} = c^{(j)}, \qquad C^{(j)(-)} = C^{(j)} + c^{(j)} \tag{107b}$$

with $c^{(j)}$ being scalar constants, which imply via (104) that the two constant $D \times D$ matrices $C^{(j)}$ which appear in these evolution equations satisfy the symmetry conditions

$$C^{(j)} = -(-)^j S C^{(j)T} S, \qquad j = 0, 1, \tag{107c}$$

and the auxiliary dependent variable $W \equiv W(x,t)$ (which is also a $D \times D$ matrix) satisfy the symmetry condition

$$W = -S W^T S. \tag{107d}$$

The constant $D \times D$ matrix S appearing in these evolution equations is diagonal and made up of (*a priori* arbitrary) signs:

$$S = \mathrm{diag}[s_1, \ldots, s_D], \qquad s_k^2 = 1, \qquad k = 1, \ldots, D, \tag{107e}$$

and moreover, without significant loss of generality, we can assume the constant $D \times D$ matrix $C^{(1)}$ to be traceless,

$$\mathrm{Trace}[C^{(1)}] = 0 \tag{107f}$$

(of course $C^{(0)}$ is as well traceless, since all its diagonal elements vanish, see (107c) with (107e)).

Let us point out that these integrable PDEs (106) should be compared with (48), with which they coincide if the vector $\vec{u}(x,t)$ is *real*, $\vec{u}(x,t) = \vec{u}^*(x,t)$.

For $D = 2$, via the assignments

$$\vec{u}(x,t) = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix}, \qquad W(x,t) = \begin{pmatrix} 0 & -s_1 w(x,t) \\ s_2 w(x,t) & 0 \end{pmatrix}, \tag{108a}$$

$$S = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}, \qquad C^{(0)} = \begin{pmatrix} 0 & s_1 \tilde{a} \\ -s_2 \tilde{a} & 0 \end{pmatrix}, \qquad C^{(1)} = \begin{pmatrix} -b & s_1 \tilde{b} \\ s_2 \tilde{b} & b \end{pmatrix}, \tag{108b}$$

which clearly satisfy the symmetry requirements (107) (without entailing any additional restriction on the constants $\tilde{a}$, b and $\tilde{b}$), these evolution equations (106) take the following form:

$$u_{1,t} - s_1 \tilde{a} u_2 - b u_{1,x} + s_1 \tilde{b} u_{2,x} = -s_1 w u_2, \tag{109a}$$

$$u_{2,t} + s_2 \tilde{a}^* u_1 + b u_{2,x} + s_2 \tilde{b} u_{1,x} = s_2 w u_1, \tag{109b}$$

$$w_x = 2 b s_1 s_2 u_1 u_2 + \tilde{b}\left(s_1 u_1^2 - s_2 u_2^2\right), \tag{109c}$$

with the latter equation supplemented by the boundary conditions

$$w(-\infty, t) = 0. \tag{109d}$$

Provided $\tilde{a}$ is *real* and *nonvanishing*, $\tilde{a} \neq 0$, the signs s_1 and s_2 are equal so that $s_1 s_2 = +$, and b and $\tilde{b}$ are as well *real* and *do not both vanish*, this system is easily seen to be dispersive, namely to possess (in the linear limit obtained by setting $w(x,t) = 0$) the solution

$$u_j(x,t) = A_j \exp\{\mathrm{i}[kx - \omega(k)t]\}, \qquad j = 1, 2, \tag{110a}$$

with the 'relativistic' dispersion relation

$$\omega(k) = \pm\sqrt{(b^2 + \tilde{b}^2)k^2 + \tilde{a}^2}, \tag{110b}$$

which is then clearly real for all real values of k.

Let us note that, if $\tilde{a}$ is *real*, $\tilde{a} = \tilde{a}^*$, and $\tilde{b}$ vanishes, $\tilde{b} = 0$, this integrable system of PDEs (109) coincides with (54) provided the dependent variables are also restricted to be *all real*, $u_1(x,t) = u_1^*(x,t), u_2(x,t) = u_2^*(x,t), w(x,t) = w^*(x,t)$. As a consequence the additional reduction of the system of PDEs (109) to the single scalar PDEs (66) and (69) applies as well here.

Returning to (105) we now note that in the 'square' case $N^{(+)} = N^{(-)} = M$ the following *additional* reduction,

$$S^{(\pm)} = s_\pm \mathbf{1}, \qquad C^{(0)(\pm)} = C^{(0)}, \qquad C^{(1)(\pm)} = \mp C^{(1)}, \qquad W^{(\pm)}(x,t) = \mp W(x,t), \tag{111a}$$

is compatible with the conditions (104), provided the $M \times M$ matrices $C^{(0)}$ and $W(x,t)$ are *antisymmetric*, the $M \times M$ matrix $C^{(1)}$ is *symmetric*, and the $M \times M$ matrix $U(x,t)$ satisfies the symmetry property $U^T(x,t) = \tilde{s} U(x,t)$, namely it is *symmetric* if the sign $\tilde{s}$ is *positive* and instead *antisymmetric* if the sign $\tilde{s}$ is *negative*:

$$C^{(0)T} = -C^{(0)}, \qquad C^{(1)T} = C^{(1)}, \qquad U^T(x,t) = \tilde{s} U(x,t), \qquad W^T(x,t) = -W(x,t). \tag{111b}$$

The corresponding integrable system of matrix PDEs reads

$$U_t - [C^{(0)}, U] + \{C^{(1)}, U_x\} = [W, U], \tag{112a}$$

$$W_x = s[C^{(1)}, U^2], \qquad s = \tilde{s} s_+ s_-. \tag{112b}$$

For $M = 2$ this equation is nontrivial only if $\tilde{s} = +$, when via the assignments

$$U(x,t) = \begin{pmatrix} u_1(x,t) & v(x,t) \\ v(x,t) & u_2(x,t) \end{pmatrix}, \qquad W(x,t) = w(x,t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{113a}$$

$$C^{(0)} = a \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad C^{(1)} = \begin{pmatrix} b & c \\ c & -b \end{pmatrix}, \tag{113b}$$

it reads

$$u_{1t} + 2av + 2bu_{1x} + 2cv_x = -2wv, \tag{114a}$$

$$u_{2t} - 2av - 2bu_{2x} + 2cv_x = 2wv, \tag{114b}$$

$$v_t - a(u_1 - u_2) + c(u_{1x} + u_{2x}) = w(u_1 - u_2), \tag{114c}$$

$$w_x = s(u_1 + u_2)[c(u_1 - u_2) - 2bv]. \tag{114d}$$

Note that this system of PDEs coincides with the (permissible) reduction of the system (100) when one requires it to only contain *real* quantities (consistently with the identification of symmetrical matrices with Hermitian matrices in the *real* context). Moreover this system (114) is equivalent to the system (89) of one PDE and two ODEs, via the change of dependent variables

$$\psi_x = -(u_1 + u_2), \qquad \psi_t = 2b(u_1 - u_2) + 4cv, \tag{115a}$$

$$z_1 = 2(w + a), \qquad u_3 = -2c(u_1 - u_2) + 4bv, \tag{115b}$$

provided one also sets $b^2 + c^2 = 1/4$.

Note that by inserting the 'plane-wave' solution

$$v(x,t) = B\exp\{\mathrm{i}[kx - \omega(k)t]\}, \qquad u_j(x,t) = A_j\exp\{\mathrm{i}[kx - \omega(k)t]\}, \qquad j = 1, 2, \tag{116a}$$

in the linearized version of this system (114) (namely, in its first *three* PDEs, with $w(x,t) = 0$) one gets three determinations for $\omega(k)$, a trivial one, $\omega(k) = 0$, and the 'relativistic' expression

$$\omega(k) = \pm 2\sqrt{(b^2 + c^2)k^2 + a^2}. \tag{116b}$$

It is thus seen that, if the constants a is *real* and does not vanish, $a = a^* \neq 0$, and the constants b and c are such that the sum of their squares is *positive*, $b^2 + c^2 > 0$ (as is certainly the case if both these constants are *real* and at least one of them does not vanish), then this system (114) is dispersive.

Additional reductions of this integrable system of PDEs (114) are possible, but the linearized versions of these reduced models are generally *not* dispersive, and therefore (perhaps) less interesting. For instance the particular reduction

$$u_1(x,t) = u(x,t), \qquad u_2(x,t) = [u(x,t)]^*, \tag{117a}$$

which is compatible with this system provided the constants a and b are *imaginary*, the constant c is *real*, and the two dependent variables $v(x,t)$ respectively $w(x,t)$ are *real* respectively *imaginary*,

$$a = -a^*, \qquad b = -b^*, \qquad c = c^*, \qquad v(x,t) = [v(x,t)]^*, \qquad w(x,t) = -[w(x,t)]^*, \tag{117b}$$

yields the model

$$u_t + 2bu_x + 2cv_x = -2(a + w)v, \tag{118a}$$

$$v_t + 2c\,\mathrm{Re}[u_x] = 2\mathrm{i}(a + w)\,\mathrm{Im}[u], \tag{118b}$$

$$w_x = 4s\,\mathrm{Re}[u]\{\mathrm{i}c\,\mathrm{Im}[u] - bv\}, \tag{118c}$$

which models wave propagation only if $|c| > |b|$ and $a = 0$ (see (116b) with (117b)).

The last reduction of (112) we report is characterized by $M = 3$ and $s_- = -$, so that the 3×3 matrix $U(x,t)$ is *antisymmetric*, as well as $W(x,t)$ and $C^{(0)}$, see (111b). Hence in this case each of these three matrices can be conveniently parameterized via the three components of a 3-vector, according to the standard assignment

$$U(x,t) = \begin{pmatrix} 0 & u_3(x,t) & -u_2(x,t) \\ -u_3(x,t) & 0 & u_1(x,t) \\ u_2(x,t) & -u_1(x,t) & 0 \end{pmatrix}, \tag{119a}$$

$$W(x,t) = \begin{pmatrix} 0 & -z_3(x,t) & z_2(x,t) \\ z_3(x,t) & 0 & -z_1(x,t) \\ -z_2(x,t) & z_1(x,t) & 0 \end{pmatrix}, \tag{119b}$$

$$C^{(0)} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}. \tag{119c}$$

We moreover assume the symmetric 3×3 matrix $C^{(1)}$ to be *dyadic*, hence also parameterized by the three components of a 3-vector,

$$C^{(1)} = \vec{b}\vec{b}^T = \begin{pmatrix} b_1^2 & b_1 b_2 & b_1 b_3 \\ b_1 b_2 & b_2^2 & b_2 b_3 \\ b_1 b_3 & b_2 b_3 & b_3^2 \end{pmatrix}. \tag{119d}$$

This last assignment is more restrictive than it would be necessary in order to guarantee the symmetric character of $C^{(1)}$, but it allows us to express the integrable system of coupled PDEs (112) in the following neatly *covariant* form:

$$\vec{u}_t = \vec{a} \wedge \vec{u} + (\vec{b} \cdot \vec{u}_x)\vec{b} - (\vec{b} \cdot \vec{b})\vec{u}_x + \vec{z} \wedge \vec{u}, \tag{120a}$$

$$\vec{z}_x = s(\vec{b} \cdot \vec{u})\vec{b} \wedge \vec{u}, \tag{120b}$$

where $s = -s_+$ is an arbitrarily assignable sign, and we assume as usual that the ODE (120*b*) is supplemented by the boundary condition

$$\vec{z}(-\infty, t) = 0. \tag{120c}$$

Note that these equations are *real*, and therefore may possess *real* solutions, if the 3-vector $\vec{a}$ is *real*, $\vec{a} = \vec{a}^*$, and the 3-vector $\vec{b}$ is either *real* or *imaginary*, $\vec{b} = \vec{b}^*$ or $\vec{b} = -\vec{b}^*$.

The dispersion relation associated with the linearized version of this system is obtained of course by requiring that the plane wave $\vec{u}(x, t) = \vec{A} \exp\{\mathrm{i}[kx - \omega(k)t]\}$ satisfies the linear part of the first (120*a*) of these two PDEs, namely this PDE (120*a*) with $\vec{z}(x, t) = 0$. Its three branches correspond to the three roots of the cubic equation

$$\omega^3 - 2b^2 k\omega^2 + (b^4 k^2 - a^2)\omega + [a^2 b^2 - (\vec{a} \cdot \vec{b})^2]k = 0, \tag{121}$$

where $a^2 \equiv \vec{a} \cdot \vec{a}$ and $b^2 \equiv \vec{b} \cdot \vec{b}$. Two special cases are worth noticing. (i) If the two 3-vectors $\vec{a}$ and $\vec{b}$ are *orthogonal*, $\vec{a} \cdot \vec{b} = 0$, these three determinations of $\omega(k)$ read

$$\omega(k) = b^2 k, \qquad \omega(k) = \tfrac{1}{2} b^2 k \pm \sqrt{\tfrac{1}{4} b^4 k^2 + a^2}. \tag{122}$$

Hence in this case the linearized system is dispersive, provided the squared modulus $a^2 \equiv \vec{a} \cdot \vec{a}$ of the 3-vector $\vec{a}$ is *positive* and the squared modulus $b^2 \equiv \vec{b} \cdot \vec{b}$ of the 3-vector $\vec{b}$ is either *real* or *imaginary* (which is certainly the case if the 3-vector $\vec{a}$ is *real* and it does not vanish, $\vec{a} = \vec{a}^* \neq \vec{0}$, and the 3-vector $\vec{b}$ is either *real* or *imaginary*, $\vec{b} = \vec{b}^*$ or $\vec{b} = -\vec{b}^*$, when the integrable system (120) admits real solutions, see above). (ii) If the two 3-vectors $\vec{a}$ and $\vec{b}$ are *parallel*, so that $a^2 b^2 - (\vec{a} \cdot \vec{b})^2 = 0$, the three determinations of $\omega(k)$ read

$$\omega(k) = 0, \qquad \omega(k) = b^2 k \pm a, \tag{123}$$

hence in this case the linearized system is *not dispersive*.

Let us now consider the special case of the system (120) in which the two 3-vectors $\vec{a}$ and $\vec{b}$ are orthogonal, $\vec{a} \cdot \vec{b} = 0$. We then introduce the three orthogonal unit vectors $\hat{a}$, $\hat{b}$ and $\hat{c}$ by setting

$$\vec{a} = a\hat{a}, \qquad \vec{b} = b\hat{b}, \qquad \hat{c} = \hat{a} \wedge \hat{b}, \tag{124}$$

and represent the two 3-vectors $\vec{u}(x, t)$ and $\vec{z}(x, t)$ via their components along these unit vectors:

$$\vec{u} = u_1\hat{a} + u_2\hat{b} + u_3\hat{c}, \qquad \vec{z} = z_1\hat{a} + z_3\hat{c}. \tag{125}$$

Note that $\vec{z}(x,t)$ has no component in the $\hat{b}$ direction, as implied by (120*b*) with (120*c*). The system (120) takes thereby the following form:

$$u_{1t} = -b^2 u_{1x} - u_2 z_3, \tag{126a}$$

$$u_{2t} = -a u_3 + u_1 z_3 - u_3 z_1, \tag{126b}$$

$$u_{3t} = a u_2 - b^2 u_{3x} + u_2 z_1, \tag{126c}$$

$$z_{1x} = s b^2 u_2 u_3, \qquad z_{3x} = -s b^2 u_1 u_2. \tag{126d}$$

Note that, if $a = 0$, one can introduce an additional reduction (clearly compatible with these equations) by setting

$$u_3(x,t) = z_1(x,t) = 0, \tag{127a}$$

or alternatively

$$u_1(x,t) = u_3(x,t), \qquad z_1(x,t) = -z_3(x,t), \tag{127b}$$

whereby this system (126) reduces again (with an appropriate choice of the parameters k_n, ω_n and g_n) to the system of three first-order PDEs (57).

3. Single-soliton solutions: boomerons and trappons

The 'single-soliton' solution for the general matrix evolution equation (24) with the boundary condition (26*a*) can be easily obtained via standard techniques from the Lax pair given in the next section, or from the results of section 3 of [5]. It reads as follows:

$$Q^{(\pm)}(x,t) = \frac{p\exp(\pm \mathrm{i}kx)A^{(\pm)}(t)}{\cosh\{p[x-\xi(t)]\}}, \tag{128a}$$

$$W^{(\pm)}(x,t) = p(\mathrm{sign}[\mathrm{Re}(p)] + \tanh\{p[x-\xi(t)]\})[C^{(1)(\pm)}, A^{(\pm)}(t)A^{(\mp)}(t)], \tag{128b}$$

with p and k being scalar constants (x- and t-independent, possibly *complex*—but hereafter we restrict for simplicity attention to the case in which these numbers are both *real* and *positive*, entailing of course that sign $[\mathrm{Re}(p)] = 1$), and with $\xi(t)$ the single scalar function that (if *real*, as it is the case for the simple cases to which our attention is hereafter confined) clearly identifies, see (128) and below, the position of the soliton as it evolves over time. The two 'amplitude' matrices $A^{(\pm)}(t)$ are constrained by the scalar condition

$$\mathrm{Trace}[A^{(+)}(t)A^{(-)}(t)] = -1. \tag{129}$$

Moreover, in the cases considered below the two matrices $A^{(\pm)}(t)$ are *dyadic*, and the vectors that characterize their dyadic structure (see below) may be interpreted as those that identify the (time-dependent) 'polarization' of the soliton as it evolves over time. Note however that in some cases a simple dyadic structure of the two matrices $A^{(\pm)}(t)$ is incompatible with the reduction under consideration, entailing a more complex structure of the 'single-soliton' solution.

The function $\xi(t)$ is given by the following formula:

$$\xi(t) = \xi(0) + \rho(t), \tag{130a}$$

$$\rho(t) = (2p)^{-1}\log\left[\frac{f(0)}{f(t)}\right], \tag{130b}$$

$$f(t) = \left(b^{(+)(-)T}(0)\exp\left(-\Gamma_+^{(-)}t\right)\exp\left(\Gamma_-^{(-)}t\right)b^{(-)(-)}(0)\right)$$
$$\times\left(b^{(-)(+)T}(0)\exp\left(-\Gamma_-^{(+)}t\right)\exp\left(\Gamma_+^{(+)}t\right)b^{(+)(+)}(0)\right), \qquad (130c)$$

$$\Gamma_s^{(s')} = C^{(0)(s')} + (\mathrm{i}k + sp)C^{(1)(s')}, \qquad s = \pm, \qquad s' = \pm. \qquad (130d)$$

Here $C^{(j)(+)}$, respectively $C^{(j)(-)}$, are of course the $N^{(+)} \times N^{(+)}$, respectively $N^{(-)} \times N^{(-)}$, constant square matrices that appear in the matrix evolution equations (24), while the two (column) $N^{(+)}$-vectors $b^{(\pm)(+)}(0)$ respectively the *two* (column) $N^{(-)}$-vectors $b^{(\pm)(-)}(0)$ characterize the 'initial polarization' matrices $A^{(\pm)}(0)$ of the soliton via the following dyadic relations (valid for all time, and in particular at $t = 0$):

$$A^{(+)}(t) = b^{(+)(+)}(t)b^{(+)(-)T}(t), \qquad A^{(-)}(t) = b^{(-)(-)}(t)b^{(-)(+)T}(t). \qquad (131)$$

These vectors $b^{(s)(s')}(t)$, $b^{(s)(+)}(t)$ respectively $b^{(s)(-)}(t)$ being $N^{(+)}$-dimensional respectively $N^{(-)}$-dimensional, are restricted by the (single, scalar) 'normalization condition' (valid for all time!)

$$(b^{(+)(-)T}b^{(-)(-)})(b^{(-)(+)T}b^{(+)(+)}) = -1 \qquad (132)$$

(see (129) and (131)). Let us re-emphasize that in the above equations, and throughout, the symbol T denotes transposition, hence it transforms column vectors into row vectors, and of course the (scalar) product of a transposed vector (on the left) times a vector (on the right) is a *scalar*, while the product of a vector (on the left) times a transposed vector (on the right) is a *dyadic* matrix.

To complete our display of the single-soliton solution (128) we must exhibit the time-evolution of the two (dyadic matrix) amplitudes $A^{(\pm)}(t)$ (where $A^{(+)}(t)$ respectively $A^{(-)}(t)$ is of course a dyadic $N^{(+)} \times N^{(-)}$, respectively $N^{(-)} \times N^{(+)}$, matrix), or equivalently, see (131), of the four 'polarization vectors' $b^{(s)(s')}(t)$. It reads (see (130*d*))

$$b^{(s)(s)}(t) = \exp\left[\tfrac{1}{2}p\rho(t) + \Gamma_s^{(s)}t\right]b^{(s)(s)}(0), \qquad s = \pm, \qquad (133a)$$

$$b^{(s)(-s)}(t) = \exp\left[\tfrac{1}{2}p\rho(t) - \Gamma_s^{(-s)T}t\right]b^{(s)(-s)}(0), \qquad s = \pm, \qquad (133b)$$

of course with $\rho(t)$ defined by (130*b*) with (130*c*) and the matrices $\Gamma_s^{(s')}$ defined by (130*d*). These expressions imply that the normalization condition (132) remains true for all time if it holds at the initial time $t = 0$ (this is not quite obvious, but is in fact true, as can be easily verified).

Let us now tersely discuss the asymptotic behaviour of this single soliton solution as the space coordinate x diverges to the left and to the right, $x \to \mp\infty$, and also its asymptotic behaviour in the remote past and future, namely as $t \to \mp\infty$.

The behaviour as the space coordinate x diverges, which characterizes the *shape* of this solution, is immediately evident from the formula (128): the components $Q^{(\pm)}(x, t)$ are localized ('solitonic' shape), namely they vanish asymptotically, $Q^{(\pm)}(\pm\infty, t) = 0$, while the components $W^{(\pm)}(x, t)$ have a 'kink-like' shape, in particular they vanish to the left, $W^{(\pm)}(-\infty, t) = 0$, and tend instead to a finite (generally time-dependent) value to the right,

$$W^{(\pm)}(\infty, t) = 2p[C^{(1)(\pm)}, A^{(\pm)}(t)A^{(\mp)}(t)]. \qquad (133c)$$

The asymptotic behaviour in time requires a somewhat more detailed discussion, which entails going, up to minor variants, over developments already reported in [5]. Throughout this discussion we assume for simplicity that both the matrices $\Gamma_s^{(s')}$, see (130*d*), and the 'initial data' characterizing this single soliton solution, are *generic*. In particular we assume that the matrices $\Gamma_s^{(s')}$ are *diagonalizable* and that their eigenvalues $\gamma_s^{(s')(n)}$, and the corresponding eigenvectors $\chi_s^{(s')(n)}$,

$$\Gamma_s^{(s)}\chi_s^{(s)(n)} = \gamma_s^{(s)(n)}\chi_s^{(s)(n)}, \qquad n = 1, \ldots, N^{(s)}, \qquad s = \pm, \tag{134a}$$

$$\Gamma_{-s}^{(s)T}\chi_{-s}^{(s)(n)} = \gamma_{-s}^{(s)(n)}\chi_{-s}^{(s)(n)}, \qquad n = 1, \ldots, N^{(s)}, \qquad s = \pm, \tag{134b}$$

can be ordered so that

$$\mathrm{Re}\big[\gamma_s^{(s')(1)}\big] < \mathrm{Re}\big[\gamma_s^{(s')(n)}\big] < \mathrm{Re}\big[\gamma_s^{(s')(N^{(s')})}\big], \qquad n = 2, \ldots, N^{(s')} - 1, \quad s = \pm, \quad s' = \pm, \tag{135}$$

entailing a unique definition of the *first* one, respectively the *last* one, of these eigenvalues, as being characterized by having the *smallest*, respectively the *largest*, real part. In the following, without significant loss of generality (since the dependence on the trace of these matrices can be factored out via a trivial transformation of type $Q^{(\pm)}(x, t) \to Q^{(\pm)}(x, t)\exp(at + bx)$ with a and b appropriate *scalar* constants), we restrict for simplicity attention to matrices $C^{(0)(s)}$ and $C^{(1)(s)}$, $s = \pm$, that are *traceless*, implying that the matrices $\Gamma_s^{(s')}$, see (130*d*), are as well traceless,

$$\mathrm{Trace}\big[\Gamma_s^{(s')}\big] = 0, \tag{136}$$

and this condition, together with the ordering convention (135), of course entails that the *first* eigenvalue has a *negative* real part and the *last* one a *positive* real part,

$$\mathrm{Re}\big[\gamma_s^{(s')(1)}\big] < 0, \qquad \mathrm{Re}\big[\gamma_s^{(s')(N^{(s')})}\big] > 0, \qquad s = \pm, \qquad s' = \pm. \tag{137}$$

It is then easy to see that the following formulae characterize the asymptotic behaviour of the 'position' $\mathrm{Re}[\xi(t)]$ and 'amplitude' $A^{(\pm)}(t)$ that enter in the single-soliton solution (128):

$$\xi(t) \underset{\to\pm\infty}{\to} \xi_\pm + V_\pm t, \tag{138a}$$

$$\xi_\pm = \xi(0) + \frac{1}{2p}\log\left[\frac{\pm\gamma_\pm}{2p\beta_\pm}\right], \tag{138b}$$

$$V_\pm = \mp\frac{\gamma_\pm}{2p}, \tag{138c}$$

$$A^{(s)}(t) \underset{\to\pm\infty}{\to} \exp(\mathrm{i}s\mu_\pm t)\bar{A}_\pm^{(s)}, \qquad s = \pm, \tag{139a}$$

$$\bar{A}_\pm^{(s)} = \left[\frac{\pm\gamma_\pm}{2p\beta_\pm}\right]^{1/2}\big(\chi_s^{(\mp s)(1)T}b^{(s)(\mp s)}(0)\big)\big(\chi_s^{(\pm s)(N^{(\pm s)})T}b^{(s)(\pm s)}(0)\big)P_\pm^{(s)}, \qquad s = \pm, \tag{139b}$$

$$P_+^{(s)} = \chi_s^{(s)(N^{(s)})}\chi_s^{(-s)(1)T}, \qquad P_-^{(s)} = \chi_s^{(s)(1)}\chi_s^{(-s)(N^{(-s)})T}, \qquad s = \pm. \tag{139c}$$

Here we used the shorthand notations

$$\gamma_s = \gamma_s^{(+)(N^{(+)})} + \gamma_{-s}^{(-)(N^{(-)})} - \gamma_{-s}^{(+)(1)} - \gamma_s^{(-)(1)}, s = \pm, \tag{140}$$

$$\beta_s = (\chi_-^{(s)(1)T}b^{(-)(s)}(0))\big(\chi_+^{(-s)(1)T}b^{(+)(-s)}(0)\big)\cdot \\ \times\big(\chi_+^{(s)(N^{(s)})T}b^{(+)(s)}(0)\big)(\chi_-^{(-s)(N^{(-s)})T}b^{(-)(-s)}(0))\cdot \\ \times\sum_{s'=\pm}s'\big(\chi_{-s's}^{(s')(1)T}\chi_{s's}^{(s')(N^{(s')})}\big)\big(\chi_{s's}^{(-s')(1)T}C_s^{(1)(-s')}\chi_{-s's}^{(-s')(N^{(-s')})}\big), \\ s = \pm, \tag{141}$$

with (to compactify the notation) $C_+^{(1)(-s')} \equiv C^{(1)(-s')}$ and $C_-^{(1)(-s')} \equiv C^{(1)(-s')T}$, and

$$\mu_s = \frac{\mathrm{i}}{2}\left[-\gamma_s^{(+)(N^{(+)})} + \gamma_{-s}^{(-)(N^{(-)})} - \gamma_{-s}^{(+)(1)} + \gamma_s^{(-)(1)}\right], \qquad s = \pm. \tag{142}$$

We are of course assuming that the 'initial data' for this single-soliton solution are as well *generic*, entailing that none of the scalar products appearing in the right-hand sides of (139*b*) and (141) vanish.

Let us end this section with two remarks.

Remark 3.1. Via the linear coordinate transformation (21) the general form of the evolution equation (24) becomes

$$\mathrm{d}Q_{\tilde{t}}^{(\pm)} \mp c\left[C^{(1)(\pm)}Q_{\tilde{t}}^{(\pm)} - Q_{\tilde{t}}^{(\pm)}C^{(1)(\mp)}\right] + bQ_{\tilde{x}}^{(\pm)} \mp a\left[C^{(1)(\pm)}Q_{\tilde{x}}^{(\pm)} - Q_{\tilde{x}}^{(\pm)}C^{(1)(\mp)}\right]$$
$$- [C^{(0)(\pm)}Q^{(\pm)} - Q^{(\pm)}C^{(0)(\mp)}] = \mp[W^{(\pm)}Q^{(\pm)} + Q^{(\pm)}W^{(\mp)}], \tag{143a}$$

$$cW_{\tilde{t}}^{(\pm)} + aW_{\tilde{x}}^{(\pm)} = [C^{(1)(\pm)}, Q^{(\pm)}Q^{(\mp)}]. \tag{143b}$$

The above remark is mathematically rather trivial, but it allows one to capture a larger set of evolution equations of possible *applicative* relevance: note that the general structure of this system of ODEs (143) is always characterized by the presence of (partial) derivatives of *first-order* only, both with respect to the (new) time variable $\tilde{t}$ and to the (new) space variable $\tilde{x}$. Of course an analogous transformation can be applied to *all* the *integrable* systems of nonlinear PDEs obtained from the general matrix PDE (24) by reductions, see the preceding section 2; their explicit display is left as an easy exercise for the diligent reader.

Remark 3.2. Of course the single-soliton solution appropriate for the integrable class of nonlinear matrix PDEs (143) are obtained from the formulae written above, see (128) and the relations following it, via the independent-variable transformation (21). This entails that the analysis of the shape of the single-soliton solution, as a function of the (new) space variable $\tilde{x}$ (for fixed $\tilde{t}$), is then (partially) modified: in particular it is clear that, while the conclusions about the localized character of the components $Q^{(\pm)}$ considered as functions of $\tilde{x}$ at fixed $\tilde{t}$ are generically unchanged ($Q^{(\pm)}$ vanishes at both spatial ends, namely as $\tilde{x} \to \pm\infty$), it is now clear from (128*b*), (138*a*) and (21) that a necessary condition to guarantee that the components $W^{(\pm)}$ be as well *localized* (namely, vanish asymptotically at *both* ends, as $\tilde{x} \to \pm\infty$ for fixed $\tilde{t}$) is (see (138*a*)) that $c \neq 0$, namely that also $|t| \longrightarrow \infty$ as $\tilde{x} \to \pm\infty$, entailing that in the limit the matrices $A^{(\pm)}(t)$ get replaced by their asymptotic expressions (139). The localization of $W^{(\pm)}(x, t)$ is then entailed by the additional condition that the commutators $\left[C^{(1)(s)}, P_\pm^{(s)}P_\pm^{(-s)}\right]$ vanish. A condition sufficient to guarantee that this happens is the commutativity of the two matrices $C^{(0)(s)}$ and $C^{(1)(s)}$ (which however may also cause the disappearance of the boomeronic effect).

These two remarks are important inasmuch as they demonstrate that the *integrable* class of (systems of) evolution PDEs considered in this paper include equations the single-soliton solutions of which may have some components which are localized while other components are not localized (they are kink-like), as well as single-soliton solutions that feature instead components which are *all* localized.

The discussion of the adaptation of these results—whenever possible—to the reduced equations discussed in section 2 is for the moment left as an instructive exercise for the diligent reader.

4. The Lax pair

The Lax pair which underlies the *integrable* character of the (systems of) nonlinear PDEs discussed in this paper is easily obtained from that given in [5]. It reads, in self-evident notation,

$$\Psi_x(x,t;\lambda) = [\lambda\sigma + Q(x,t)]\Psi(x,t;\lambda), \tag{144a}$$
$$\Psi_t(x,t;\lambda) = \{C^{(0)} - \sigma W(x,t) + \sigma[C^{(1)}, Q(x,t)] + 2\lambda C^{(1)}\}\Psi(x,t;\lambda), \tag{144b}$$

where of course Ψ is an $N \times N$ matrix and λ is the spectral parameter (a scalar). It is indeed easy to verify that precisely the system of $N \times N$ matrix nonlinear PDEs (10) is yielded by the requirement that this Lax pair (144) satisfies, for all values of the spectral parameter λ, the compatibility condition

$$\Psi_{tx}(x,t;\lambda) = \Psi_{xt}(x,t;\lambda), \tag{145}$$

namely by the requirement that the partial t-derivative of the right-hand side of (144a) equals the partial x-derivative of the right-hand side of (144b).

5. Outlook

As already mentioned above, several standard additional results can be obtained for all the integrable (systems of) nonlinear PDEs identified in this paper. This shall eventually be done by us and/or by others, especially for those instances of these systems that turn out to have *applicative* relevance. To make progress towards identifying such instances we plan to revisit soon, if need be also in the context of nonlinear *matrix* PDEs, the multiscale expansion technique (see, for instance, [7]) that provides a convenient avenue to identify equations that possess both properties, to be *integrable* and to be *widely applicable* [1].

References

[1] Calogero F 1990 Why are certain nonlinear PDEs both widely applicable and integrable? *What is Integrability?* ed V E Zakharov (Berlin: Springer) pp 1–62

[2] Calogero F and Degasperis A 1976 Nonlinear evolution equations solvable by the inverse spectral transform associated with the multichannel Schrödinger problem, and properties of their solutions *Lett. Nuovo Cimento* **15** 65–9

Calogero F and Degasperis A 1976 Coupled nonlinear evolution equations solvable via the inverse spectral transform and solitons that come back: the boomeron *Lett. Nuovo Cimento* **16** 425–33

Calogero F and Degasperis A 1976 Bäcklund transformations, nonlinear superposition principle, multisoliton solutions and conserved quantities for the 'boomeron' nonlinear evolution equation *Lett. Nuovo Cimento* **16** 434–8

Calogero F and Degasperis A 1977 Nonlinear evolution equations solvable by the inverse spectral transform. II *Nuovo Cimento* B **39** 1–54

Degasperis A 1978 Solitons, boomerons, trappons *Nonlinear Evolution Equations Solvable by the Spectral Transform* ed F Calogero (London: Pitman) pp 97–126

[3] Calogero F and Degasperis A 1980 Nonlinear evolution equations solvable by the inverse spectral transform associated to the matrix Schrödinger equation *Solitons* (*Topics in Current Physics* vol 17) ed R K Bullough and P J Caudrey (Berlin: Springer) pp 301–24

[4] Calogero F and Degasperis A 1982 *Spectral Transform and Solitons* (Amsterdam: North-Holland)

[5] Calogero F and Degasperis A 2004 New integrable equations of nonlinear Schrödinger type *Stud. Appl. Math.* **113** 91–137

[6] Calogero F and Degasperis A 2005 Novel solution of the system describing the resonant interaction of three waves *Physica* D **200** 242–56

[7] Calogero F, Degasperis A and Xiaoda Ji 2000 Nonlinear Schrödinger-type equations from multiscale reduction of PDEs. I. Systematic derivation *J. Math. Phys.* **41** 6399–443

[8] Degasperis A and Lombardo S 2006 Exact solutions of the 3-wave resonant interaction equation *Physica* D **214** 157–68

[9] Degasperis A, Rogers C and Schief W 2002 Isothermic surfaces generated by Bäcklund transformations: boomeron and zoomeron connections *Stud. Appl. Math.* **109** 39–65

[10] Kaup D J, Reiman A and Bers A 1979 Space-time evolution of nonlinear three-wave interactions. I. Interaction in a homogeneous medium *Rev. Mod. Phys.* **51** 275–309

[11] Zakharov V E and Manakov S V 1973 Resonant interaction of wave packets in non-linear media *JETP Lett.* **18** 243–5

Zakharov V E and Manakov S V 1975 *Zh. Eksp. Teor. Fiz.* **69** 1654–73

Zakharov V E and Manakov S V 1976 The theory of resonant interaction of wave packets in nonlinear media *Sov. Phys.—JETP* **42** 842–50 (Engl. Transl.)